

Correlators with $s\ell_2$ Yangian symmetry

J. Fuksa^{a,c,1}, R. Kirschner^{b,2}

^a *Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, Dubna, Russia*

^b *Institut für Theoretische Physik, Universität Leipzig,
PF 100 920, D-04009 Leipzig, Germany*

^c *Faculty of Nuclear Sciences and Physical Engineering,
Czech Technical University in Prague, Czech republic*

Abstract

Correlators based on $s\ell_2$ Yangian symmetry and its quantum deformation are studied. Symmetric integral operators can be defined with such correlators as kernels. Yang-Baxter operators can be represented in this way. Particular Yangian symmetric correlators are related to the kernels of QCD parton evolution. The solution of the eigenvalue problem of Yangian symmetric operators is described.

¹E-MAIL: FUKSA@THEOR.JINR.RU

²E-MAIL: ROLAND.KIRSCHNER@ITP.UNI-LEIPZIG.DE

1 Introduction

The symmetry on which quantum integrable systems (QIS) like spin chains are based can be formulated in terms of a Yangian algebra [1], which results from the expansion of the monodromy matrix. In the applications of QIS to gauge field theories, which attracted much attention in the last two decades, the symmetry appears typically in infinite-dimensional representations with action on functions. The first of such applications concerned the high-energy asymptotics of scattering in Quantum Chromodynamics and are based on sl_2 [2, 3]. In the Regge asymptotics [4] the dynamics reduces to the plane transverse to the scattering axis and in the Bjorken asymptotics [5, 6] the dynamics reduces to a light ray. The latter asymptotics is connected with the scale dependence of composite operators. In the case of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory the relation to QIS has been shown [7] to work with the super $sl(4|4)$ spin chain for all composite operators in all orders of perturbative expansion and to be connected to strings via the AdS/CFT correspondence, where in the supergravity on the string side classical integrable systems appear. The advances in the methods of scattering amplitude computations have lead in particular for the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory to the discovery of the dual superconformal symmetry [8] and the understanding that the two superconformal symmetries are embedded in the corresponding Yangian algebra [9].

Yangian symmetric correlators (YSC) have been proposed as a convenient formulation of the Yangian symmetry of scattering amplitudes [10]. They are defined by an eigenvalue relation involving the monodromy matrix operator. A number of properties, previously observed in amplitude calculations, have been shown to follow from this monodromy condition. There are convenient methods of construction of YSC, the most important one is based on Yang-Baxter *RLL* type relations. The construction methods are related to the on-shell graph method developed in the amplitude context. This relation has been discussed in detail in [11].

The tool of YSC is not restricted in its application to super Yang-Mills scattering amplitudes. It may be used wherever Yangian symmetry appears, it is powerful and it will help to exploit the higher symmetry in more problems. The investigation of the particular case of sl_2 is important for testing the approach, first of all because this case is relevant in a number of integrable models, in two-dimensional conformal field theories [12] and in particular questions of gauge field theory. The advantage of the method should be tested here in comparison with the standard methods, reproducing known results in alternative ways and going beyond them by relying on the new viewpoint provided. The second reason for devoting attention to the sl_2 case is the fact that here Yang-Baxter relations are known in several versions and in most explicit form.

In the present paper we expand the range for Yangian symmetric correlators. We shall show that the approach of YSC is powerful also beyond the investigation of super Yang-Mills scattering amplitudes, where it has been introduced. We shall demonstrate its flexibility by indicating several ways of calculation, several explicit formulations, the opportunity to perform algebraic deformations.

In examples for the sl_2 case we shall demonstrate new construction methods beyond the ones known in the amplitude context and the role of the spectral parameters characterizing the representation of the Yangian algebra.

The approach allows to treat the Yangian symmetry in a convenient way. We shall discuss the role of YSC as kernels of symmetric operators. The symmetry of the correlators induces the symmetry of the integral operators.

The approach has a high potential of physical applications. We shall show how the Yangian symmetry of the correlators leads to the exact solution of the spectral problem for the related operators, which is of great importance in view of physics. Among many applications related to the sl_2 case we pick up the example of the scale dependence of parton distributions and composite operators. We shall establish a direct relation of the kernels of the leading order scale evolution to YSC.

In Sect. 2 we recall the definition of YSC and formulations of Yang-Baxter relations. We use the explicit expressions of Yang-Baxter (YB) operators together with methods developed for amplitudes [13] to obtain explicit expressions of YSC. We emphasize the specifics of the case of sl_2 , which allows several versions in the construction in particular those differing essentially from the ones close to the amplitude methods. A number of explicit examples will be given for illustration in Sect. 3.

We keep arbitrary spectral parameters in the monodromy matrix from which the YSC construction starts, explain their relation to the dilatation weight associated with each point and also how they characterize the representation of the Yangian algebra. Note that in $\mathcal{N} = 4$ super Yang-Mills amplitudes the case of vanishing spectral parameters matters and therefore the case of arbitrary values has been considered in a few papers only, e.g. [14, 11, 15, 16]. We shall see that spectral parameters play an active role in the method.

The symmetry condition defining YSC generalizes to algebraic deformations. We demonstrate this in the trigonometric case in Sect. 4.

In Sect. 5 we study Yangian symmetric operators defined as integral operators with YSC as kernels. We show how particular 4-point YSC result in the Yang-Baxter operators. Generalized YB operators result from particular $2M$ -point YSC.

It is known how YB relations allow to derive the spectrum of the involved YB R operator. In Sect. 6 we show that this works also for generalized YB operators by demonstrating it in the simplest non-trivial case with $M = 3$.

The application of QIS and Yangian symmetry to the scale dependence of composite operators and of (generalized) parton distributions [17] is well known in the formulation of quantum spin chain dynamics. In Sect. 5 we present the formulation in terms of YSC. We identify the kernels of the scale evolution of (generalized) parton distributions as 4-point YSC with particular values of the dilatation weights and the spectral parameters. In the presented form of ratio-coordinates interpreted as positions on the light ray these kernels describe the scale dependence of operators composed of two fields with derivatives in the two-point light ray form.

The generalized Yang-Baxter operators constructed from $2M$ -point YSC are relevant for the scale evolution of operators composed out of M fields. Their spectra, the calculation of which we show in the particular case $M = 3$, allows to derive exactly the related anomalous dimensions. In the situation of a low number of fields this approach becomes an alternative to the Bethe ansatz calculations used in the spin chain formulation.

2 Yangian symmetry

2.1 Monodromy matrix operators and symmetric correlators

We start with the formulation of the sl_2 Lie algebra relations as the fundamental Yang-Baxter relation

$$\mathcal{R}(u-v)(L(u) \otimes I)(I \otimes L(v)) = (I \otimes L(v))(L(u) \otimes I)\mathcal{R}(u-v). \quad (2.1)$$

Here $\mathcal{R}(u-v)$ stands for Yang's R matrix, which is 4×4 in our case and composed of the unit matrix and the matrix representing the permutation of the tensor factors in the tensor product of the two-dimensional fundamental representation spaces, $\mathcal{R}(u) = uI_{4 \times 4} + P$. The L matrices are linear in the spectral parameter u , $L(u) = uI_{2 \times 2} + L$. With this ansatz (2.1) implies that the matrix elements of L generate the sl_2 algebra. We shall introduce several forms of L below.

The Yangian algebra can be introduced by the matrix operator $T(u)$ obeying the above YB relation with $L(u)$ substituted by $T(u)$ and where now the dependence on u is not restricted as above. Then the generators of the Yangian algebra [1] appear in the expansion of $T(u)$ in inverse powers of u and the Yangian algebra relations follow from the fundamental YB relation (2.1). We shall deal with evaluations of the sl_2 Yangian of finite order N where $T(u)$ is constructed from the L matrices in the way well know in the context of integrable models. Let the matrix elements of $L_i(u_i)$ act on the representation space V_i and consider the matrix product defining the monodromy matrix

$$T(u, \delta_1, \dots, \delta_N) = T(\mathbf{u}) = L_1(u_1) \cdots L_N(u_N), \quad u_i = u + \delta_i,$$

acting on the tensor product $V_1 \otimes \cdots \otimes V_N$. In the case if the representations V_i are of definite weight $2\ell_i$ the weight parameters are added to the parameter list abbreviated by \mathbf{u} . We consider infinite-dimensional representations V_i in terms of functions of two or one variables. In the case of V_i of definite weight the functions of two variables $\mathbf{x}_i = (x_{i,1}, x_{i,2})$ are homogeneous and the weight $2\ell_i$ is the degree of homogeneity,

$$\psi(\mathbf{x}_i) = x_{i,2}^{2\ell_i} \phi(x_i) = x_{i,1}^{2\ell_i} \tilde{\phi}(y_i).$$

The one-dimensional variables are related to the two-dimensional as ratios of the latter components, where we have two versions in the following discussion, $x_i = \frac{x_{i,1}}{x_{i,2}}$ and $y_i = -\frac{x_{i,2}}{x_{i,1}}$. The elements of the tensor product $V_1 \otimes \cdots \otimes V_N$ are then functions of N points, two- or one-dimensional.

N -point functions Φ obeying the condition

$$T(\mathbf{u})\Phi = IE(\mathbf{u})\Phi \tag{2.2}$$

are called Yangian symmetric correlators (YSC). On l.h.s an operator-valued matrix is acting and on r.h.s. there is the unit matrix I and the eigenvalue depending on the set of parameters \mathbf{u} . As solutions we shall accept also expressions involving distributions, extending the representation space $V_1 \otimes \cdots \otimes V_N$. The δ -distributions are to be understood in Dolbeault sense (cf. [18]), their arguments must not be restricted to real values.

The eigenvalue $E(\mathbf{u})$ is related to the quantum determinant of the monodromy $T(\mathbf{u})$. In the sl_n case the definition is

$$\text{qdet}(T(\mathbf{u})) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) T_{1\sigma(1)}(u) T_{2\sigma(2)}(u-1) \cdots T_{n\sigma(n)}(u-n+1),$$

where the sum is performed over the symmetric group S_n of the order n . On r.h.s. $T_{\alpha\beta}(u-k)$ are the matrix elements of $T(u, \delta_1, \dots, \delta_N)$. The quantum determinant $\text{qdet}(T(\mathbf{u}))$ is the generating function for the center of Yangian $Y(gl_n)$. Because the off-diagonal elements of $T(\mathbf{u})$ annihilate the function Φ it follows immediately from (2.2) that in our case of $n=2$

$$\text{qdet}(T(\mathbf{u}))\Phi = T_{11}(u)T_{22}(u-1)\Phi = E(u)E(u-1)\Phi.$$

2.2 The homogeneous form

We start with two forms of the L operators, $L_i^\pm(u) = Iu + L_i^\pm(0)$, with matrix elements

$$L_{i,\alpha\beta}^+(0) = \partial_{i,\alpha} x_{i,\beta}, \quad L_{i,\alpha\beta}^-(0) = -x_{i,\alpha} \partial_{i,\beta} \quad (2.3)$$

built from two canonical pairs $\mathbf{x}_i, \mathbf{p}_i$ at each point i , $\mathbf{x}_i = (x_{i,1}, x_{i,2})$, $\mathbf{p}_i = (\partial_{i,1}, \partial_{i,2})$. Notice that the elementary canonical transformation

$$\begin{pmatrix} x_{i,\alpha} \\ \partial_{i,\alpha} \end{pmatrix} \rightarrow \begin{pmatrix} \partial_{i,\alpha} \\ -x_{i,\alpha} \end{pmatrix} \quad (2.4)$$

transforms $L_i^+(u)$ into $L_i^-(u)$. Here the transformation acts uniformly for the index values $\alpha = 1, 2$. With a transformation modified in this respect, i.e. non-trivial for the index value $\alpha = 1$ only,

$$\begin{pmatrix} x_{i,1} \\ \partial_{i,1} \end{pmatrix} \rightarrow \begin{pmatrix} \partial_i^\lambda \\ -\lambda_i \end{pmatrix}, \quad \begin{pmatrix} x_{i,2} \\ \partial_{i,2} \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\lambda}_i \\ \bar{\partial}_i^\lambda \end{pmatrix}, \quad (2.5)$$

applied to $L_i^+(u)$ we are lead to the helicity form¹

$$L_i^\lambda(0) = \begin{pmatrix} -\lambda_i \partial_i^\lambda & -\lambda_i \bar{\lambda}_i \\ \partial_i^\lambda \bar{\partial}_i^\lambda & \bar{\partial}_i^\lambda \lambda_i \end{pmatrix}.$$

We have the matrix inversion relations

$$\begin{aligned} \left(\frac{L_i^+(u)}{u} \right)^{-1} &= \frac{L_i^+(-u-1-(\mathbf{x}_i \mathbf{p}_i))}{-u-1-(\mathbf{x}_i \mathbf{p}_i)}, \\ \left(\frac{L_i^-(u)}{u} \right)^{-1} &= \frac{L_i^-(-u+1+(\mathbf{x}_i \mathbf{p}_i))}{-u+1+(\mathbf{x}_i \mathbf{p}_i)}, \end{aligned} \quad (2.6)$$

where $(\mathbf{x}_i \mathbf{p}_i) = x_{i,1} \partial_{i,1} + x_{i,2} \partial_{i,2}$. We work with representations on functions of $\mathbf{x}_i = (x_{i,1}, x_{i,2})$ or $\lambda, \bar{\lambda}$ so that $\mathbf{p}_i = (\partial_{i,1}, \partial_{i,2})$ or $\partial_i^\lambda, \bar{\partial}_i^\lambda$ act as derivatives as suggested by notation.

We consider the YB relations of RLL type

$$\begin{aligned} R_{12}^{++}(u_1 - u_2) L_1^+(u_1) L_2^+(u_2) &= L_1^+(u_2) L_2^+(u_1) R_{12}^{++}(u_1 - u_2), \\ R_{12}^{+-}(u_1 - u_2) L_1^+(u_1) L_2^-(u_2) &= L_1^+(u_2) L_2^-(u_1) R_{12}^{+-}(u_1 - u_2), \\ R_{12}^{\lambda\lambda}(u_1 - u_2) L_1^\lambda(u_1) L_2^\lambda(u_2) &= L_1^\lambda(u_2) L_2^\lambda(u_1) R_{12}^{\lambda\lambda}(u_1 - u_2), \end{aligned} \quad (2.7)$$

and find the following expressions for the YB operators:

$$R_{12}^{+-}(u) = (\mathbf{x}_1 \cdot \mathbf{x}_2)^u, \quad R_{12}^{++}(u) = \int \frac{dc}{c^{1+u}} e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)}. \quad (2.8)$$

Here the integration contour is arbitrary with the restriction that integration by parts is performed with vanishing boundary terms (Appendix A). The expression for $R_{12}^{\lambda\lambda}$ is obtained from the one for $R_{12}^{++}(u)$ by substituting according to the canonical transformation (2.5)

$$(\mathbf{x}_1 \cdot \mathbf{p}_2) \rightarrow -\partial_1^\lambda \lambda_2 + \bar{\lambda}_1 \bar{\partial}_2^\lambda.$$

¹We use here the term helicity form in analogy to the case of sl_4 Yangian symmetry, where it has been applied to scattering amplitudes.

If the representation spaces V_1, V_2 are restricted to the subspaces of eigenvalues $2\ell_i$ of $(\mathbf{x}_i \mathbf{p}_i)$ then the inversion relations (2.6) imply further Yang-Baxter relations. The dependence on the eigenvalue of the restricted L matrices is conveniently formulated by a second argument $u^+ = u + 2\ell$ or $u^- = u - 2\ell - 2$ as

$$L^+(u)|_{2\ell} = L^+(u^+, u), \quad L^-(u)|_{2\ell} = L^-(u, u^-).$$

The additional YB relations read

$$R_{21}^{+-}(u_1^- - u_2^+) L_1^-(u_1, u_1^-) L_2^+(u_2^+, u_2) = L_1^-(u_1, u_2^+) L_2^+(u_1^-, u_2) R_{21}^{+-}(u_1^- - u_2^+), \quad (2.9)$$

$$R_{21}^{++}(u_1^+ - u_2^+) L_1^+(u_1^+, u_1) L_2^+(u_2^+, u_2) = L_1^+(u_2^+, u_1) L_2^+(u_1^+, u_2) R_{21}^{++}(u_1^+ - u_2^+).$$

The previous relations read in the new form

$$R_{12}^{+-}(u_1 - u_2) L_1^+(u_1^+, u_1) L_2^-(u_2, u_2^-) = L_1^+(u_1^+, u_2) L_2^-(u_1, u_2^-) R_{12}^{+-}(u_1 - u_2), \quad (2.10)$$

$$R_{12}^{++}(u_1 - u_2) L_1^+(u_1^+, u_1) L_2^+(u_2^+, u_2) = L_1^+(u_1^+, u_2) L_2^+(u_2^+, u_1) R_{12}^{++}(u_1 - u_2).$$

Each of the relations is characterized by the exchange of a pair of parameters. We shall consider the transformation of monodromy matrices as $T(\mathbf{u})R = RT(\mathbf{u}')$ accompanied by a change of the parameters by permutations $\sigma, \bar{\sigma}$ of the sets u_1, \dots, u_N and u_1^+, \dots, u_N^+ . This is conveniently written as the table in two rows

$$\begin{pmatrix} u_{\sigma(1)} & u_{\sigma(2)} & \dots & u_{\sigma(N)} \\ u_{\bar{\sigma}(1)}^+ & u_{\bar{\sigma}(2)}^+ & \dots & u_{\bar{\sigma}(N)}^+ \end{pmatrix}.$$

We shall use also the abbreviation by writing the indices carried by the parameters only.

2.3 The ratio-coordinate form

The reduction of the L operators to the subspace of eigenvalue 2ℓ is done for the action on functions of the coordinate components by the restriction to a definite degree of homogeneity.

$$\begin{aligned} \psi_{2\ell}(\mathbf{x}) &= (x_2)^{2\ell} \phi(x), \quad L^+(u)(x_2)^{2\ell} \phi(x) = (x_2)^{2\ell} L_x(u^+ + 1, u) \phi(x), \\ \psi_{2\ell}(\mathbf{x}) &= (x_1)^{2\ell} \tilde{\phi}(y), \quad L^-(u)(x_1)^{2\ell} \tilde{\phi}(y) = (x_1)^{2\ell} L_y(u, u^- + 1) \tilde{\phi}(y), \end{aligned} \quad (2.11)$$

$$x = \frac{x_1}{x_2}, \quad y = -\frac{x_2}{x_1}.$$

The conjugated momenta are to be substituted in the case $+$ as

$$x_1 \partial_1 \rightarrow x \partial, \quad x_2 \partial_2 = (\mathbf{x} \mathbf{p}) - x_1 \partial_1 \rightarrow 2\ell - x \partial,$$

and in the case $-$ as

$$x_1 \partial_1 = (\mathbf{x} \mathbf{p}) - x_2 \partial_2 \rightarrow 2\ell - y \partial, \quad x_2 \partial_2 \rightarrow y \partial.$$

Explicitly we have

$$L^+(u) = \begin{pmatrix} u + 1 + x_1 \partial_1 & x_2 \partial_1 \\ x_1 \partial_2 & u + 1 + x_2 \partial_2 \end{pmatrix},$$

$$L_x(u^+ + 1, u) = \begin{pmatrix} u + 1 + x\partial & \partial \\ x(-x\partial + 2\ell) & u + 1 + 2\ell - x\partial \end{pmatrix},$$

$$L^-(u) = \begin{pmatrix} u - x_1\partial_1 & -x_1\partial_2 \\ -x_2\partial_1 & u - x_2\partial_2 \end{pmatrix}, \quad L_y(u, u^- + 1) = \begin{pmatrix} u - 2\ell + y\partial & \partial \\ y(-y\partial + 2\ell) & u - y\partial \end{pmatrix}.$$

In this way the L operators in the two dual representations of gl_2 result in the same form of L operators of sl_2 , $L(u, u^- + 1) = L(\tilde{u}^+ + 1, \tilde{u})$ for $\tilde{u} = u^- + 1 = u - 2\ell - 1$. Therefore we can omit the superscripts \pm in the ratio-coordinate form. The uniformization of both results is achieved also in the notation used in [26]. We substitute for the signature $+$ case $v = u + \ell$, but for the signature $-$ case $v = u - \ell - 1$. Then the reduced L appears in both cases with the same arguments, $L(v^{(1)} + 1, v^{(2)} + 1)$, where

$$v^{(1)} = v + \ell, v^{(2)} = v - \ell - 1. \quad (2.12)$$

The matrix $L(v^{(1)}, v^{(2)})$ factorizes as

$$L(v^{(1)}, v^{(2)}) = \begin{pmatrix} v^{(2)} + 1 + x\partial & \partial \\ x(-x\partial + 2\ell) & v^{(1)} - x\partial \end{pmatrix} = v^{(1)}\hat{V}^{-1}(v^{(1)})\hat{D}\hat{V}(v^{(2)}), \quad (2.13)$$

$$\hat{V}(v) = \begin{pmatrix} v & 0 \\ -x & -1 \end{pmatrix}, \quad \hat{D} = \begin{pmatrix} 1 & -\partial \\ 0 & 1 \end{pmatrix}.$$

This is one argument in favor of the choice of the parameters v with the particular shift with respect to the original ones u . The other (related) argument is the permutation action discussed below.

The L matrix and the sl_2 generators are related as

$$L(v^{(1)}, v^{(2)}) = \begin{pmatrix} v + S^0 & S^- \\ S^+ & v - S^0 \end{pmatrix},$$

$$S^- = \partial, \quad S^+ = x(-x\partial + 2\ell), \quad S^0 = x\partial - \ell. \quad (2.14)$$

The inversion relations (2.6) both read in the notation $L(v^{(1)}, v^{(2)})$

$$L^{-1}(v^{(1)}, v^{(2)}) = -\frac{1}{v^{(1)}v^{(2)}}L(-v^{(2)}, -v^{(1)}). \quad (2.15)$$

This can also be checked directly starting from the factorized form (2.13).

The YB relations formulated above (2.7), (2.9), (2.10) can be rewritten in the ratio coordinates with the notation $L(v^{(1)}, v^{(2)})$ which does not distinguish signature. The resulting forms of the YB relations are characterized by particular permutations of the parameters $v_1^{(1)}, v_1^{(2)}, v_2^{(1)}, v_2^{(2)}$ appearing as arguments of the L operators in the product $L_1(v_1^{(1)}, v_1^{(2)})L_2(v_2^{(1)}, v_2^{(2)})$, [25].

$$R_{12}^1(v_1^{(1)}|v_2^{(1)}, v_2^{(2)})L_1(v_1^{(1)}, v_1^{(2)})L_2(v_2^{(1)}, v_2^{(2)}) = L_1(v_2^{(1)}, v_1^{(2)})L_2(v_1^{(1)}, v_2^{(2)})R_{12}^1(v_1^{(1)}|v_2^{(1)}, v_2^{(2)}),$$

$$R_{12}^2(v_1^{(1)}, v_1^{(2)}|v_2^{(2)})L_1(v_1^{(1)}, v_1^{(2)})L_2(v_2^{(1)}, v_2^{(2)}) = L_1(v_1^{(1)}, v_2^{(2)})L_2(v_2^{(1)}, v_1^{(2)})R_{12}^2(v_1^{(1)}, v_1^{(2)}|v_2^{(2)}).$$

It is appropriate to consider the YB operators related to the elementary permutations [26],

$$\mathcal{S}_{11}(v_1^{(1)} - v_1^{(2)})L_1(v_1^{(1)}, v_1^{(2)})L_2(v_2^{(1)}, v_2^{(2)}) = L_1(v_1^{(2)}, v_1^{(1)})L_2(v_2^{(1)}, v_2^{(2)})\mathcal{S}_{11}(v_1^{(1)} - v_1^{(2)}),$$

$$\begin{aligned}\mathcal{S}_{12}(v_1^{(2)} - v_2^{(1)})L_1(v_1^{(1)}, v_1^{(2)})L_2(v_2^{(1)}, v_2^{(2)}) &= L_1(v_1^{(1)}, v_2^{(1)})L_2(v_1^{(2)}, v_2^{(2)})\mathcal{S}_{12}(v_1^{(2)} - v_2^{(1)}), \\ \mathcal{S}_{22}(v_2^{(1)} - v_2^{(2)})L_1(v_1^{(1)}, v_1^{(2)})L_2(v_2^{(1)}, v_2^{(2)}) &= L_1(v_1^{(1)}, v_1^{(2)})L_2(v_2^{(2)}, v_2^{(1)})\mathcal{S}_{22}(v_2^{(1)} - v_2^{(2)}).\end{aligned}$$

In the RLL relations the R operators R^1 and R^2 resulting from the above $R_{12}^{++}(u_1 - u_2)$ and $R_{21}^{++}(u_1^+ - u_2^+)$ interchange the parameters with superscript 1 or 2 respectively. The YB operator $\mathcal{S}_{12}(v_1^{(2)} - v_2^{(1)})$ interchanging the adjacent $v_1^{(2)}, v_2^{(1)}$ results from R_{12}^{+-} or R_{21}^{+-} . Using (2.13) it is obtained as

$$\mathcal{S}_{12}(v) = (x_1 - x_2)^v. \quad (2.16)$$

Notice that here x_i are the ratio coordinates and the subscript is the point label. The other elementary permutations of $v_1^{(1)}, v_1^{(2)}$ or of $v_2^{(1)}, v_2^{(2)}$ appear in the RLL relations with $\mathcal{S}_{11}(v_1^{(1)} - v_1^{(2)})$ or $\mathcal{S}_{22}(v_2^{(1)} - v_2^{(2)})$, respectively. They are given by the intertwiner between the representations of weights 2ℓ and $-2\ell - 2$.

$$\begin{aligned}\mathcal{S}_{11}(v_1^{(1)} - v_1^{(2)}) &= W_1(2\ell_1 + 1), \quad \mathcal{S}_{22}(v_2^{(1)} - v_2^{(2)}) = W_2(2\ell_2 + 1), \\ W_i(a) &= x_i^{-a} \frac{\Gamma(x_i \partial_i + 1)}{\Gamma(x_i \partial_i + 1 - a)}.\end{aligned} \quad (2.17)$$

The R operators related to the permutations of parameters of superscript 1 or 2 factorize into the operators \mathcal{S}_{ij} related to the elementary permutations.

$$\begin{aligned}R_{12}^1(u^{(1)}|v^{(1)}, v^{(2)}) &= \mathcal{S}_{12}(v^{(2)} - v^{(1)})\mathcal{S}_{22}(u^{(1)} - v^{(1)})\mathcal{S}_{12}(u^{(1)} - v^{(2)}), \\ R_{12}^2(u^{(1)}, u^{(2)}|v^{(2)}) &= \mathcal{S}_{12}(u^{(2)} - u^{(1)})\mathcal{S}_{11}(u^{(2)} - v^{(2)})\mathcal{S}_{12}(u^{(1)} - v^{(2)}).\end{aligned}$$

This leads to the expressions

$$\begin{aligned}R_{12}^1(u^{(1)}|v^{(1)}, v^{(2)}) &= \frac{\Gamma(x_{21}\partial_2 + u^{(1)} - v^{(2)} + 1)}{\Gamma(x_{21}\partial_2 + v^{(1)} - v^{(2)} + 1)}, \\ R_{12}^2(u^{(1)}, u^{(2)}|v^{(2)}) &= \frac{\Gamma(x_{12}\partial_1 + u^{(1)} - v^{(2)} + 1)}{\Gamma(x_{12}\partial_1 + u^{(1)} - u^{(2)} + 1)},\end{aligned}$$

where $x_{12} = x_1 - x_2$ and $x_{21} = x_2 - x_1$.

Whereas the action of the operator $\mathcal{S}_{12}(u)$ on functions of x_1, x_2 is multiplicative, the action of the latter operators and of $\mathcal{S}_{11}, \mathcal{S}_{22}$ can be defined with the help of the Beta integral representation and $\hat{N} = x\partial$. We have

$$W(2\ell + 1) = \frac{x^{-2\ell-1}}{\Gamma(-2\ell-1)} B(\hat{N} + 1, -2\ell - 1) = \text{const} \frac{1}{x^{2\ell+1}} \int ds s^{\hat{N}} (1-s)^{-2\ell-2}.$$

The action on functions can be defined (up to a constant factor) as

$$W(2\ell + 1)f(x) = \frac{1}{x^{2\ell+1}} \int ds (1-s)^{-2\ell-2} f(sx) = \int d\tilde{x} (x - \tilde{x})^{-2\ell-2} f(\tilde{x}). \quad (2.18)$$

This is clear by $s^{x\partial}x = xs^{x\partial+1} = (sx)s^{x\partial}$.

We shall consider the transformation of monodromy matrices like $T(\mathbf{v})R = RT(\mathbf{v}')$ accompanied by permutations $\tilde{\sigma}$ of the set of N parameter pairs $\mathbf{v} = v_1^{(1)}, v_1^{(2)}; \dots; v_N^{(1)}, v_N^{(2)}$. The resulting parameter set is now conveniently presented as one string of N pairs

$$v_{\sigma(1^{(1)})}, v_{\sigma(1^{(2)})}; \dots; v_{\sigma(N^{(1)})}, v_{\sigma(N^{(2)})}.$$

Restricting the L operator in the helicity form to a subspace of a particular helicity eigenvalue $2h$ of $-\partial^\lambda \lambda + \bar{\lambda} \bar{\partial}^\lambda$ related to the coordinate dilatation weight 2ℓ as $2h = 2\ell + 1$, we may introduce another form of the L operator involving only one canonical pair. We change from $\lambda, \bar{\lambda}$ to k, l and calculate the corresponding conjugated operators from

$$\bar{\lambda}^2 = kl, \quad \lambda^2 = \frac{k}{l}, \quad \lambda \partial^\lambda = k \partial_k - l \partial_l, \quad \bar{\lambda} \bar{\partial}^\lambda = k \partial_k + l \partial_l.$$

We restrict the functions of $\lambda, \bar{\lambda}$ to the form $\psi(\lambda, \bar{\lambda}) = l^h \phi(k)$.

$$L^\lambda(u) \cdot l^h \phi(k) = l^h L^k(u + h, h) \cdot \phi(k)$$

and obtain $L^k(u, h) = Iu + L^k(0, h)$,

$$L^k(0, h) = \begin{pmatrix} -k \partial_k & -k \\ \frac{1}{k}(k \partial_k + h)(k \partial_k - h) & \partial_k k \end{pmatrix}.$$

The result is to be compared with the ratio coordinate reduction $L(v^{(1)}, v^{(2)})$ after the canonical transformation $x, \partial \rightarrow \partial_k, -k$. If the ratio coordinate x describes position then k plays the role of momentum.

We add the remark that the factor carrying the weight can be chosen differently, e.g., $\psi(\lambda, \bar{\lambda}) = l^h k^a \phi'(k)$. Then the equivalent representation of the algebra on $\phi'(k)$ is generated by $L^{k'}(0, h) = k^{-a} L^k(0, h) k^a$.

2.4 Constructions of symmetric correlators

Any transformation of the monodromy matrix $T \rightarrow T'$ induces a map from a solution Φ of the symmetry condition (2.2) with T to a solution Φ' of the symmetry condition with T' . Examples of such transformations have been discussed in [10]. They are related to features observed in the construction of scattering amplitudes in gauge field theories [13].

If we separate the set of points into two subsets $1, \dots, N \rightarrow 1, \dots, M; M+1, \dots, N$ and correspondingly factorize the monodromy matrix $T_{1, \dots, N} = T_{1, \dots, M} T_{M+1, \dots, N}$ the symmetric correlator $\Phi^{(1)}(\mathbf{x}_1, \dots, \mathbf{x}_M)$ of the condition (2.2) with $T_{1, \dots, M}$ and the symmetric correlator $\Phi^{(2)}(\mathbf{x}_{M+1}, \dots, \mathbf{x}_N)$ of the condition (2.2) with $T_{M+1, \dots, N}$ results in a solution of the condition (2.2) with the full monodromy matrix $T_{1, \dots, N}$ simply by product, $\Phi = \Phi^{(1)}(\mathbf{x}_1, \dots, \mathbf{x}_M) \Phi^{(2)}(\mathbf{x}_{M+1}, \dots, \mathbf{x}_N)$.

We find immediately elementary correlators solving the symmetry condition in the case $N = 1$ where the monodromy consists of one factor L only.

$$L^+(u) \cdot 1 = I(u+1), \quad L^-(u) \cdot 1 = Iu,$$

$$L^+(u) \cdot \delta^{(2)}(\mathbf{x}) = Iu \delta^{(2)}(\mathbf{x}), \quad L^-(u) \cdot \delta^{(2)}(\mathbf{x}) = I(u+1) \delta^{(2)}(\mathbf{x}). \quad (2.19)$$

In the first case the weight is $2\ell = 0$, in the second $2\ell = -2$.

$$L^\lambda(u) \cdot \delta(\lambda) = I(u+1) \delta(\lambda), \quad L^\lambda(u) \cdot \delta(\bar{\lambda}) = Iu \delta(\bar{\lambda}). \quad (2.20)$$

Here the weights are again 0 and -2 , correspondingly. The weights are calculated as eigenvalues of $(\mathbf{x} \cdot \mathbf{p}) = \bar{\lambda} \bar{\partial}^\lambda - \partial^\lambda \lambda$. In ratio coordinates we have

$$L(v, v-1) \cdot 1 = Iv \quad (2.21)$$

In the last case the weight is zero and therefore the two arguments are related as $v^{(1)} - v^{(2)} = 1$.

Starting from a solution of the Yangian symmetry condition (2.2) other solutions can be generated by action on Φ with YB operators obeying particular RLL relations with L operators entering the monodromy. This can be implemented both in the formulations with homogeneous coordinates or with coordinate ratios. It is an important tool of constructing YSC and it will be illustrated in the following section.

We act with a sequence of R operators on a trivial correlator composed as a product of elementary 1-point correlators given above. The R operators, in particular the points on which they act non-trivially and their arguments, have to be chosen such that the RLL relations allow to permute them with the product of L operators of the monodromy matrix. In this way we obtain that the particular sequence of R operations applied to the initial trivial correlator produces another Yangian symmetric correlator corresponding to the monodromy resulting by the permutations from the initial monodromy.

In the form (2.14) the generators correspond to infinitesimal Moebius transformations, where the finite transformation can be written as $x \rightarrow x' = \frac{ax+b}{cx+d}$, $ad - bc = 1$, in the particular case of vanishing weight, $2\ell = 0$. The conformal transformations acting on functions as

$$F(x_1, \dots, x_N) \rightarrow \prod (cx_i + d)^{2\ell_i} F(x'_1, \dots, x'_N)$$

are generated by $\sum S_i^a$. Conformal symmetric correlators are invariant under this transformation and thus obey the condition

$$\sum_1^N S_i^a \cdot \Phi(x_1, \dots, x_N) = 0. \quad (2.22)$$

The general form of N -point functions obeying (2.22) is well known. Beyond $N = 3$ the condition admits an arbitrary dependence on anharmonic ratios like $\frac{x_{12}x_{34}}{x_{13}x_{24}}$. The Yangian symmetry condition (2.2) implies (2.22). Moreover, it fixes the freedom left by the global conformal symmetry.

3 Examples of Yangian symmetric correlators

If working in the homogeneous coordinate form each R^{++} operator action comes with a contour integral ((2.8), Appendix A). The resulting multiple integral can be rewritten in the standard link integral form with the integration variables being coordinates of the related Grassmannian. This step is not specific to sl_2 and the resulting expressions coincide with corresponding ones for amplitude calculations (sl_4) [13]. The specifics of the sl_2 case appears in the next steps to obtain the explicit forms in homogeneous, helicity or ratio coordinates. We have also the alternative to work with the R operators factors (2.16) in the ratio coordinate form directly.

3.1 Two-point correlators

We start with two-point correlators obtained from a trivial one by just one R operation. This can be formulated in homogeneous coordinates with R^{++} (2.8), in helicity coordinates with R^λ , in homogeneous coordinates with R^{+-} or directly in ratio coordinates with S_{12} (2.16).

Homogeneous coordinates

The two-point YSC with the monodromy $T_2 = L_1^+(u_2^+, u_1)L_2^+(u_1^+, u_2)$, where $u_1^+ = u_1 - 2$ and $u_2^+ = u_2$, is

$$\Phi^{-+}(u_1, u_2) = R_{21}^{++}(u_1^+ - u_2^+) \cdot \delta^{(2)}(\mathbf{x}_1) = \int \frac{dc}{c^{1+u_1^+-u_2^+}} \delta^{(2)}(\mathbf{x}_1 - c\mathbf{x}_2).$$

The monodromy results from the initial one $T_2^{(0)} = L_1^+(u_1^+, u_1)L_2^+(u_2^+, u_2)$ by the permutation of the first parameters $u_1^+ = u_1 - 2, u_2^+ = u_2$. This is written conveniently by the permutation pattern

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

We do the integration with one of the two delta distributions and transform the other one.

$$\Phi^{-+}(u_1, u_2) = \left(\frac{x_{2,2}}{x_{1,2}} \right)^{1+u_1^+-u_2^+} \delta(\langle 12 \rangle) = \left(\frac{x_{2,2}}{x_{1,2}} \right)^{1+2\ell_2} \frac{1}{x_{1,2}x_{2,2}} \delta(x_1 - x_2).$$

We use the notation

$$\langle 12 \rangle = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}, \quad x_1 = \frac{x_{1,1}}{x_{1,2}}, \quad x_2 = \frac{x_{2,1}}{x_{2,2}}.$$

We express the exponents in terms of the weights

$$2\ell_1 = u_2^+ - u_1, \quad 2\ell_2 = u_1^+ - u_2, \quad 2\ell_1 + 2\ell_2 + 2 = 0$$

and obtain

$$\Phi^{-+}(u_1, u_2) = x_{1,2}^{2\ell_1} x_{2,2}^{2\ell_2} \delta(x_1 - x_2).$$

The weights appear as degrees of homogeneity as expected.

Helicity coordinates

With the same monodromy now written in L^λ we have the correlator in the helicity form

$$\begin{aligned} \Phi^{-+}(u_1, u_2) &= R^\lambda(u_1^+ - u_2^+) \cdot \delta(\bar{\lambda}_1) \delta(\lambda_2) = \int \frac{dc}{c^{1+u_1^+-u_2^+}} \delta(\bar{\lambda}_1 - c\bar{\lambda}_2) \delta(\lambda_2 + c\lambda_1) = \\ &= \bar{\lambda}_1^{1+2\ell_1} \bar{\lambda}_2^{1+2\ell_2} \delta(\lambda_1 \bar{\lambda}_1 + \lambda_2 \bar{\lambda}_2). \end{aligned}$$

The result appears as a prefactor with exponents $2h = 2\ell + 1$ and a scale independent factor.

Ratio coordinates

Another two-point correlator with the monodromy $T_2 = L_1^+(u_1^+, u_2)L_2^-(u_1, u_2^-)$, where $u_1^+ = u_1$ and $u_2^- = u_2 - 2$, is

$$\tilde{\Phi} = R_{12}^{+-}(u_1 - u_2) \cdot 1 = (\mathbf{x}_1 \mathbf{x}_2)^{u_1 - u_2} = (x_{1,2} \ x_{2,1})^{u_1 - u_2} (x_1 - x_2)^{u_1 - u_2}.$$

The second factor in the last form is the two-point correlator obtained directly in the framework of the ratio-coordinate form by the action of \mathcal{S}_{12} . In the notation introduced for the ratio coordinate form the initial monodromy reads here $T_2^{(0)} = L_1(v_1, v_1 - 1)L_2(v_2, v_2 - 1)$ and the action by \mathcal{S}_{12} results in the permutation of the second parameter of the first factor with the first of the second, $T_2 = L_1(v_1, v_2)L_2(v_1 - 1, v_2 - 1)$. This is conveniently written by the parameter string displaying just the sequence of arguments. Indeed, the correlator with the monodromy T_2 (characterized by the corresponding parameter string) is

$$(x_1 - x_2)^{v_1 - v_2 - 1}, \quad T_2 : v_1, v_2; v_1 - 1, v_2 - 1.$$

3.2 Three-point correlators

We present examples of three-point correlator constructions with representations in homogeneous, helicity and ratio coordinates. The direct construction in ratio coordinates by three steps of action with the elementary permutation operators \mathcal{S}_{ij} is described.

Homogeneous coordinates

The three-point YSC with the monodromy characterized by the parameter permutation pattern

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

obtained by the action with $R_{21}^{++}R_{31}^{++}$ is

$$\begin{aligned} \Phi^{-++} &= R_{21}^{++}(u_3^+ - u_2^+)R_{31}^{++}(u_1^+ - u_3^+) \cdot \delta^{(2)}(\mathbf{x}_1) = \int d^2c \varphi_3 \delta^{(2)}(\mathbf{x}_1 - c_{12}\mathbf{x}_2 - c_{13}\mathbf{x}_3), \\ \varphi_3^{-1} &= c_{12}^{1+u_3-u_2} c_{13}^{1+u_1-u_3}. \end{aligned}$$

There are two delta distributions with the arguments

$$x_{1,1} - c_{12}x_{2,1} - c_{13}x_{3,1} = x'_{1,1}, \quad x_{1,2} - c_{12}x_{2,2} - c_{13}x_{3,2} = x'_{1,2}.$$

The equations $x'_{1,1} = 0$, $x'_{1,2} = 0$ are solved by

$$c_{12}^{(0)} = \frac{\langle 13 \rangle}{\langle 23 \rangle}, \quad c_{13}^{(0)} = \frac{\langle 12 \rangle}{\langle 32 \rangle},$$

and the Jacobi determinant is

$$\frac{\partial(x'_{1,1}, x'_{1,2})}{\partial(c_{12}, c_{13})} = \langle 23 \rangle.$$

From the permutation pattern we express the exponents in terms of the weights,

$$2\ell_1 = u_2 - u_1, \quad 2\ell_2 = u_3 - u_2, \quad 2\ell_1 + 2\ell_2 + 2\ell_3 + 2 = 0,$$

and obtain the result

$$\Phi^{-++} = \frac{\langle 23 \rangle^{1+2\ell_2+2\ell_3}}{\langle 13 \rangle^{1+2\ell_2} \langle 12 \rangle^{1+2\ell_3}}.$$

The symplectic products $\langle ij \rangle$ can be substituted by the differences in the ratio coordinates

$$x_{ij} = x_i - x_j, \quad x_i = \frac{x_{i,1}}{x_{i,2}},$$

$$\Phi^{-++} = x_{1,2}^{2\ell_1} x_{2,2}^{2\ell_2} x_{3,2}^{2\ell_3} \frac{x_{23}^{1+2\ell_2+2\ell_3}}{x_{13}^{1+2\ell_2} x_{12}^{1+2\ell_3}}.$$

Helicity coordinates

The three-point correlator with the same monodromy written in the helicity form in terms of L^λ is ($k_i = \lambda_i \bar{\lambda}_i$)

$$\begin{aligned} \Phi^{-++} &= \int d^2c \phi^{-++} \delta(\bar{\lambda}_1 - c_{12}\bar{\lambda}_2 - c_{13}\bar{\lambda}_3) \delta(\lambda_2 + c_{12}\lambda_1) \delta(\lambda_3 + c_{13}\lambda_1) \\ &= \left(\frac{\bar{\lambda}_1}{\lambda_1} \right)^{h_1} \left(\frac{\bar{\lambda}_2}{\lambda_2} \right)^{h_2} \left(\frac{\bar{\lambda}_3}{\lambda_3} \right)^{h_3} k_1^{-h_1} k_2^{-h_2} k_3^{-h_3} \delta(k_1 + k_2 + k_3). \end{aligned}$$

Ratio coordinates

We add examples obtained in the framework of the ratio-coordinate form. We start from the monodromy matrix in ratio coordinates and the trivial constant correlator

$$L_1(v_1, v_1 - 1)L_2(v_2, v_2 - 1)L_3(v_3, v_3 - 1) \cdot 1 = I v_1 v_2 v_3.$$

We abbreviate the monodromy matrix by the string of parameter pairs: $T_3 : v_1, v_1 - 1; v_2, v_2 - 1; v_3, v_3 - 1$. As the first step we find the symmetric correlator with the monodromy

$$\mathcal{S}_{13}(v_1^{(2)} - v_3^{(1)}) = (x_1 - x_3)^{v_1 - v_3 - 1}, \quad T_3 : v_1, v_3; v_2, v_2 - 1; v_1 - 1, v_3 - 1.$$

Because of L_2 acting on a constant results in the unit matrix the RLL relation with \mathcal{S}_{13} for the product of $L_1 L_3$ applies here. Next we apply the RLL relation with \mathcal{S}_{12} and find the correlator with the monodromy

$$(x_1 - x_3)^{v_1 - v_3 - 1} (x_1 - x_2)^{v_3 - v_2}, \quad T_3 : v_1, v_2; v_3, v_2 - 1; v_1 - 1, v_3 - 1.$$

We apply now the RLL relation with \mathcal{S}_{23} to establish the symmetric correlator with the monodromy of the parameter string

$$(x_1 - x_3)^{v_1 - v_3 - 1} (x_1 - x_2)^{v_3 - v_2} (x_2 - x_3)^{v_2 - v_1}, \quad T_3 : v_1, v_2; v_3, v_1 - 1; v_2 - 1, v_3 - 1. \quad (3.1)$$

3.3 4-point correlators

We consider again examples of construction in homogeneous, helicity and ratio coordinate form. The results will be used in the following to define symmetric integral operators and to establish relations to the QCD parton evolution.

Homogeneous coordinates

Let us consider the example

$$\begin{aligned} \Phi^{--++} &= R_{43}^{++}(u_2^+ - u_1^+) R_{32}^{++}(u_2^+ - u_4^+) R_{43}^{++}(u_1^+ - u_4^+) R_{31}^{++}(u_1^+ - u_3^+) \delta^{(2)}(\mathbf{x}_1) \delta^{(2)}(\mathbf{x}_2) \\ &= \int \frac{dc_{34}^{(2)} dc_{23} dc_{34}^{(1)} dc_{13}}{c_{34}^{(2)1+u_2^+-u_1^+} c_{23}^{1+u_2^+-u_4^+} c_{34}^{(1)1+u_1^+-u_4^+} c_{13}^{1+u_1^+-u_3^+}} \delta^{(2)}(\mathbf{x}_1 - c_{13}\mathbf{x}_3 + c_{13}(c_{34}^{(1)} + c_{34}^{(2)})\mathbf{x}_4) \\ &\quad \times \delta^{(2)}(\mathbf{x}_2 - c_{23}\mathbf{x}_3 + c_{23}c_{34}^{(2)}\mathbf{x}_4). \end{aligned}$$

The permutation pattern is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

We transform to the normal link integral form, where the integration variables coincide with the coefficients in the arguments of the δ -functions. This can be formulated in terms of the link matrices

$$\begin{pmatrix} 1 & 0 & -c_{13} & c_{13}(c_{34}^{(1)} + c_{34}^{(2)}) \\ 0 & 1 & -c_{23} & c_{23}c_{34}^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -c_{13} & -c_{14} \\ 0 & 1 & -c_{23} & -c_{24} \end{pmatrix}. \quad (3.2)$$

We obtain

$$c_{34}^{(2)} = -\frac{c_{24}}{c_{23}}, \quad c_{34}^{(1)} = \frac{c_{13}c_{24} - c_{14}c_{23}}{c_{13}c_{23}},$$

and the normal link integral form is

$$\begin{aligned}\Phi^{- - + +} &= \int d^4 c \varphi_4 \delta^{(2)}(\mathbf{x}_1 - c_{13}\mathbf{x}_3 - c_{14}\mathbf{x}_4) \delta^{(2)}(\mathbf{x}_2 - c_{23}\mathbf{x}_3 - c_{24}\mathbf{x}_4), \\ \varphi_4^{-1} &= c_{13}^{1+u_4-u_3} c_{24}^{1+u_2^+-u_1^+} (c_{13}c_{24} - c_{14}c_{23})^{1+u_1^+-u_4}.\end{aligned}\quad (3.3)$$

We have four delta distributions. The vanishing of the arguments of the first two result in

$$c_{13}^{(0)} = \frac{\langle 14 \rangle}{\langle 34 \rangle}, \quad c_{14}^{(0)} = \frac{\langle 13 \rangle}{\langle 43 \rangle},$$

and of the last two

$$c_{23}^{(0)} = \frac{\langle 24 \rangle}{\langle 34 \rangle}, \quad c_{24}^{(0)} = \frac{\langle 23 \rangle}{\langle 43 \rangle}.$$

Further,

$$c_{13}^{(0)} c_{24}^{(0)} - c_{14}^{(0)} c_{23}^{(0)} = \frac{\langle 12 \rangle}{\langle 34 \rangle}.$$

The Jacobi determinant is

$$\frac{\partial(x'_{1,1}, x'_{1,2}, x'_{2,1}, x'_{2,2})}{\partial(c_{13}, c_{14}, c_{23}, c_{24})} = \langle 34 \rangle^2.$$

The result is

$$\Phi^{- - + +} = \langle 34 \rangle^{-2} \varphi^{- - + +}(c^{(0)}) = \frac{\langle 12 \rangle^{1+u_4-u_1}}{\langle 14 \rangle^{1+u_4-u_3} \langle 23 \rangle^{1+u_2-u_1} \langle 34 \rangle^{1+u_3-u_2}}.$$

The exponents can be expressed in terms of the weights

$$2\ell_1 = u_3 - u_1, \quad 2\ell_2 = u_4 - u_2, \quad 2\ell_1 + 2\ell_3 + 2 = 0, \quad 2\ell_2 + 2\ell_4 + 2 = 0, \quad (3.4)$$

and further the weights can be replaced by the helicities $2h = 2\ell + 1$. Because only two of the weights are independent, there appears one difference in the spectral parameters additionally

$$\Phi^{- - + +} = \frac{\langle 12 \rangle^{1+2\ell_1+2\ell_2}}{\langle 14 \rangle^{1+2\ell_2} \langle 23 \rangle^{1+2\ell_1} \langle 34 \rangle} \left(\frac{\langle 12 \rangle \langle 34 \rangle}{\langle 14 \rangle \langle 23 \rangle} \right)^{u_2-u_3}.$$

The appearance of additional parameters was noticed in the amplitude context and proposed as a tool of regularization [14].

The result depends homogeneously on the components of the points i with the weights $2\ell_i$. The symplectic products $\langle ij \rangle$ can be replaced by the differences in the relative coordinates $x_{ij} = x_i - x_j$, $x_i = \frac{x_{i,1}}{x_{i,2}}$

$$\Phi^{- - + +} = x_{1,2}^{2\ell_1} x_{2,2}^{2\ell_2} x_{3,2}^{2\ell_3} x_{4,2}^{2\ell_4} \frac{x_{12}^{1+2\ell_1+2\ell_2}}{x_{14}^{1+2\ell_2} x_{23}^{1+2\ell_1} x_{34}} \left(\frac{x_{12}x_{34}}{x_{14}x_{23}} \right)^{u_2-u_3}. \quad (3.5)$$

Stepwise construction for kernels

We consider a way to an equivalent correlator where the intermediate steps have particular interpretations as kernels (see section 5.2).

$$\Phi^{- - + +} = R_{12}^{++}(u_1 - u_2) R_{21}^{++}(u_4 - u_3) R_{32}^{++}(u_2^+ - u_3) R_{41}^{++}(u_1^+ - u_4) \delta^{(2)}(\mathbf{x}_1) \delta^{(2)}(\mathbf{x}_2).$$

The first two R operations result in

$$\begin{aligned} & R_{32}^{++}(u_2^+ - u_3)R_{41}^{++}(u_1^+ - u_4)\delta^{(2)}(\mathbf{x}_1)\delta^{(2)}(\mathbf{x}_2) \\ &= \int \frac{dc_{23}dc_{14}}{c_{23}^{1+u_2^+-u_3}c_{14}^{1+u_1^+-u_4}}\delta^{(2)}(\mathbf{x}_1 - c_{14}\mathbf{x}_4)\delta^{(2)}(\mathbf{x}_2 - c_{23}\mathbf{x}_3) = \Delta^{-++}(1, 2, 3, 4). \end{aligned} \quad (3.6)$$

The monodromy is characterized by the permutation pattern

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

In the next step we have

$$\begin{aligned} R_{21}^{++}(u_4 - u_3)\Delta^{-++} &= \int \frac{d\bar{c}_{13}dc_{23}dc_{14}}{\bar{c}_{13}^{1+u_4-u_3}c_{23}^{1+u_2^+-u_4}c_{14}^{1+u_1^+-u_4}}\delta^{(2)}(\mathbf{x}_1 - \bar{c}_{13}\mathbf{x}_3 - c_{14}\mathbf{x}_4) \\ &\quad \times \delta^{(2)}(\mathbf{x}_2 - c_{23}\mathbf{x}_3), \end{aligned}$$

where \bar{c}_{13} results from a transformation of the integration variables coming with the R operators. This result is a YSC with the monodromy characterized by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}.$$

After the fourth R action the integration variables form the matrix

$$\begin{pmatrix} 1 & 0 & c_{12}c_{23} & c_{14} \\ 0 & 1 & c_{23}(1 + c_{21}c_{12}) & c_{21}c_{14} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \bar{c}_{13} & \bar{c}_{14} \\ 0 & 1 & \bar{c}_{23} & \bar{c}_{24} \end{pmatrix},$$

from which the transformation to the normal link form can be read off. We notice that the Jacobian is $c_{14}c_{23}$ and that the resulting power of \bar{c}_{14} vanishes. We obtain for the connected 4-point correlator

$$\begin{aligned} \Phi^{-++} &= \int \frac{d\bar{c}_{13}d\bar{c}_{14}d\bar{c}_{23}d\bar{c}_{24}}{\bar{c}_{13}^{1+u_4-u_3}\bar{c}_{24}^{1+u_1-u_2}(\bar{c}_{23}\bar{c}_{14} - \bar{c}_{13}\bar{c}_{24})^{1+u_2^+-u_4}}\delta^{(2)}(\mathbf{x}_1 - \bar{c}_{13}\mathbf{x}_3 - \bar{c}_{14}\mathbf{x}_4) \\ &\quad \times \delta^{(2)}(\mathbf{x}_2 - \bar{c}_{23}\mathbf{x}_3 - \bar{c}_{24}\mathbf{x}_4). \end{aligned}$$

The resulting permutation pattern allows to calculate the weights,

$$2\ell_1 = u_3 - u_2, \quad 2\ell_2 = u_4 - u_1, \quad 2\ell_1 + 2\ell_3 + 2 = 0, \quad 2\ell_2 + 2\ell_4 + 2 = 0.$$

The exponents in the result can be expressed in terms of the weights and a difference of two spectral parameters,

$$\begin{aligned} u_1 - u_2 &= 2\ell_1 - 2\ell_2 - (u_3 - u_4), \quad u_2^+ - u_4 = -2\ell_1 - 2 + (u_3 - u_4), \\ \Phi^{-++} &= x_{1,2}^{2\ell_1}x_{2,2}^{2\ell_2}x_{3,2}^{2\ell_3}x_{4,2}^{2\ell_4} \cdot \frac{x_{12}^{1+2\ell_1}}{x_{14}x_{23}^{1+2\ell_1-2\ell_2}x_{34}^{1+2\ell_2}} \left(\frac{x_{12}x_{34}}{x_{14}x_{23}} \right)^{u_4-u_3}. \end{aligned} \quad (3.7)$$

The first factors carrying the weights $2\ell_i$ are written in the second component of the homogeneous coordinates, but the last factors are written in the differences of the ratio

coordinates, i.e. $x_{12} = \frac{x_{1,1}}{x_{1,2}} - \frac{x_{2,1}}{x_{2,2}}$. Coincidence with the result of the first way (3.5) is obtained by the substitution of the parameters as

$$u_4 - u_3 \rightarrow u_2 - u_3 + 2\ell_2. \quad (3.8)$$

Helicity representation

The transformation to the helicity representation is easily done in the integral over the link variables c (3.3) by modifying the delta factors in the integrand

$$\begin{aligned} \Phi^{--++} = \int d^4c \varphi_4 \delta(\bar{\lambda}_1 - c_{13}\bar{\lambda}_3 - c_{14}\bar{\lambda}_4) \delta(\bar{\lambda}_2 - c_{23}\bar{\lambda}_3 - c_{24}\bar{\lambda}_4) \delta(\lambda_3 + c_{13}\lambda_1 + c_{23}\lambda_2) \\ \times \delta(\lambda_4 + c_{14}\lambda_1 + c_{24}\lambda_2). \end{aligned}$$

We have again four delta distributions with the arguments

$$\begin{aligned} \bar{\lambda}_1 - c_{13}\bar{\lambda}_3 - c_{14}\bar{\lambda}_4 &= \bar{\lambda}'_1, & \bar{\lambda}_2 - c_{23}\bar{\lambda}_3 - c_{24}\bar{\lambda}_4 &= \bar{\lambda}'_2, \\ \lambda_3 + c_{13}\lambda_1 + c_{23}\lambda_2 &= \lambda'_3, & \lambda_4 + c_{14}\lambda_1 + c_{24}\lambda_2 &= \lambda'_4, \end{aligned}$$

but they are not removing the four link integrations. We see the dependence between the related linear equations by multiplying the first by λ_1 , the second by λ_2 , the third by $\bar{\lambda}_3$, the fourth by $\bar{\lambda}_4$, and adding them with the result

$$\lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2 + \lambda_3\bar{\lambda}_3 + \lambda_4\bar{\lambda}_4 = 0.$$

We use the first three δ -functions to do the integrations over c_{13}, c_{14}, c_{23} . The Jacobi factor is

$$\frac{\partial(\bar{\lambda}'_1, \bar{\lambda}'_2, \lambda'_3)}{\partial(c_{13}, c_{14}, c_{23})} = \bar{\lambda}_3\bar{\lambda}_4\lambda_1.$$

Their values are fixed at

$$\begin{aligned} c_{13}^{(0)} &= \frac{\lambda_1\bar{\lambda}_1 + \lambda_4\bar{\lambda}_4 + c_{24}\bar{\lambda}_4\lambda_2}{\bar{\lambda}_3\lambda_1}, & c_{14}^{(0)} &= \frac{-\lambda_4 - c_{24}\lambda_2}{\lambda_1}, \\ c_{23}^{(0)} &= \frac{\bar{\lambda}_2 - c_{24}\bar{\lambda}_4}{\bar{\lambda}_3}, & c_{13}^{(0)}c_{24} - c_{14}^{(0)}c_{23}^{(0)} &= \frac{\bar{\lambda}_2\lambda_4 + c_{24}(\lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2)}{\bar{\lambda}_3\lambda_1}. \end{aligned}$$

We transform the fourth delta distribution by taking it together with the third one. The transformation should multiply the matrix of coefficients

$$\begin{pmatrix} c_{13} & c_{23} & 1 & 0 \\ c_{14} & c_{24} & 0 & 1 \end{pmatrix}$$

by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ \bar{\lambda}_3 & \bar{\lambda}_4 \end{pmatrix}$$

to bring the second row into the wanted form. Then we get

$$\delta(\lambda'_3)\delta(\lambda'_4) = \det(A) \delta(\lambda'_3) \delta(\lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2 + \lambda_3\bar{\lambda}_3 + \lambda_4\bar{\lambda}_4).$$

The resulting form of the correlator is

$$\Phi^{--++} = \int dc_{24} \varphi_4(c^{(0)}(c_{24})) \frac{1}{\lambda_1 \bar{\lambda}_3} \delta(\lambda_1 \bar{\lambda}_1 + \lambda_2 \bar{\lambda}_2 + \lambda_3 \bar{\lambda}_3 + \lambda_4 \bar{\lambda}_4),$$

where φ_4 is given in (3.3). We change to $c = c_{24} \lambda_2 \bar{\lambda}_4$ and $k_i = \lambda_i \bar{\lambda}_i$ in order to separate the one-dimensional momentum dependence from the homogeneity factor.

$$\begin{aligned} \Phi^{--++} &= \lambda_1^{u_1-u_3-1} \lambda_2^{u_2-u_4-1} \bar{\lambda}_3^{u_1-u_3-1} \bar{\lambda}_4^{u_2-u_4-1} \delta(k_1 + k_2 + k_3 + k_4) \\ &\quad \times \int dc c^{u_1-u_2-1} (k_1 + k_4 + c)^{u_3-u_4-1} (k_2 k_4 + c(k_1 + k_2))^{u_4-u_1+1}. \end{aligned}$$

We substitute the weights owing to the permutation pattern as in (3.4) and use the helicity notation $2h = 2\ell + 1$. Recall that we have here only two independent weights and we choose $\varepsilon = u_2 - u_3$ as the extra parameter. The integration over c is simplest in the case of vanishing momentum transfer in the channel $12 \rightarrow 34$ to which we shall refer later.

$$\begin{aligned} \Phi^{--++}|_{k_1+k_2 \rightarrow 0} &= \lambda_1^{u_1-u_3-1} \lambda_2^{u_2-u_4-1} \bar{\lambda}_3^{u_1-u_3-1} \bar{\lambda}_4^{u_2-u_4-1} \delta(k_1 + k_2 + k_3 + k_4) \\ &\quad \times \text{const} (k_1 k_3)^{2h_1+2h_2+\varepsilon-1} (k_1 - k_3)^{-2h_1-2h_2-2\varepsilon+1}. \quad (3.9) \end{aligned}$$

Ratio-coordinate construction

We add an example of construction in the ratio-coordinate form.

$$\begin{aligned} \bar{\Phi} &= \mathcal{S}_{12}(v_4^{(1)}, v_2^{(1)}) \mathcal{S}_{34}(v_3^{(2)}, v_1^{(2)}) \mathcal{S}_{23}(v_2^{(2)}, v_3^{(1)}) \mathcal{S}_{14}(v_1^{(2)}, v_4^{(1)}) \cdot 1 \\ &= x_{23}^{v_2-v_3-1} x_{14}^{v_1-v_4-1} x_{12}^{v_4-v_2} x_{34}^{v_3-v_1}. \end{aligned}$$

The resulting monodromy is characterized by the string

$$T_4 : v_1^{(1)}, v_2^{(1)}; v_4^{(1)}, v_3^{(1)}; v_2^{(2)}, v_1^{(2)}; v_3^{(2)}, v_4^{(2)}.$$

We express the parameter differences in terms of the weights, $2\ell + 1 = v^{(1)} - v^{(2)}$, and $\varepsilon = v_2 - v_3$. We have the relations $2\ell_1 + 2\ell_3 + 2 = 0$, $2\ell_2 + 2\ell_4 + 2 = 0$.

$$\bar{\Phi} = x_{14}^{2\ell_1-2\ell_2-1} x_{12}^{2\ell_2+1} x_{34}^{-2\ell_1-1} x_{23}^{-1} \left(\frac{x_{23} x_{14}}{x_{12} x_{34}} \right)^\varepsilon. \quad (3.10)$$

The result does not differ essentially from (3.5). It takes to change to the dual representation in the points 2 and 4, $\ell_2 \rightarrow \bar{\ell}_2 = -\ell_2 - 1$, $2\bar{\ell}_2 + 2\bar{\ell}_4 + 2 = 0$ and to do a cyclic shift in the point labels $1, 2, 3, 4 \rightarrow 2, 3, 4, 1$ to transform $\bar{\Phi}$ into the ratio-coordinate factor in (3.5).

3.4 A 6-point correlator

We construct a 6-point correlator appearing in subsection 5.2 as kernel of a generalised Yang-Baxter operator. We consider the construction in homogeneous coordinates only. The resulting monodromy and the relation between the spectral parameters, the dilatation weights and two extra parameters will be relevant later.

This correlator can be constructed by R operator action on $\Phi_0 = \delta^{(2)}(\mathbf{x}_1)\delta^{(2)}(\mathbf{x}_2)\delta^{(2)}(\mathbf{x}_3)$.

$$\begin{aligned}\Phi_6 &= R_{65}^{++}(u_3^+ - u_2^+)R_{54}^{++}(u_3^+ - u_1^+)R_{43}^{++}(u_3^+ - u_6)R_{65}^{++}(u_2^+ - u_1^+)R_{54}^{++}(u_2^+ - u_6) \\ &\quad \times R_{42}^{++}(u_2^+ - u_5)R_{65}^{++}(u_1^+ - u_6)R_{54}^{++}(u_1^+ - u_5)R_{41}^{++}(u_1^+ - u_4)\Phi_0 \\ &= \int d^9c \varphi_6(c) \prod_{i=1}^3 \delta^{(2)}(\mathbf{x}_i - \sum_{j=4}^6 c_{ij}\mathbf{x}_j).\end{aligned}$$

The permutation pattern is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}.$$

It allows to express the weights in terms of the spectral parameters,

$$2\ell_1 = u_4 - u_1, \quad 2\ell_2 = u_5 - u_2, \quad 2\ell_3 = u_6 - u_3.$$

The other weights are dependent by

$$2\ell_i + 2\ell_{i+3} + 2 = 0, \quad i = 1, 2, 3.$$

We express the parameters u_1, u_2, u_3 in terms of weights and then the remaining ones enter the result by the two independent parameters $\varepsilon_4 = u_4 - u_5, \varepsilon_5 = u_5 - u_6$. The monodromy is ($u = u_6$)

$$\begin{aligned}T_6 &= L_1^+(u_1)L_2^+(u_2)L_3^+(u_3)L_4^+(u_4)L_5^+(u_5)L_6^+(u_6) \\ &= L_1^+(-2\ell_1 + \varepsilon_4 + \varepsilon_5 + u)L_2^+(-2\ell_2 + \varepsilon_5 + u)L_3^+(-2\ell_3 + u)L_4^+(\varepsilon_4 + \varepsilon_5 + u)L_5^+(\varepsilon_5 + u)L_6^+(u).\end{aligned}\tag{3.11}$$

The inverse of the link variable function $\varphi_6(c)$ in the integrand is expressed in terms of the consecutive minors of the rectangular matrix

$$\begin{pmatrix} 1 & 0 & 0 & -c_{14} & -c_{15} & -c_{16} \\ 0 & 1 & 0 & -c_{24} & -c_{25} & -c_{26} \\ 0 & 0 & 1 & -c_{34} & -c_{35} & -c_{36} \end{pmatrix}.$$

denoted by triples of column numbers (ijk) . This is a feature pointed out in context of scattering amplitudes [13].

$$\begin{aligned}\varphi_6(c)^{-1} &= (234)^{1+u_5-u_4}(345)^{1+u_6-u_5}(456)^{1+u_1^+-u_6}(561)^{1+u_2^+-u_1^+}(612)^{1+u_3^+-u_2^+} \\ &= (234)^{1-\varepsilon_4}(345)^{1-\varepsilon_5}(456)^{1-2\ell_1-2+\varepsilon_4+\varepsilon_5}(561)^{1+2\ell_1-2\ell_2-\varepsilon_4}(612)^{1+2\ell_2-2\ell_3-\varepsilon_5}.\end{aligned}$$

In Appendix B we do six out of the nine integrations by the δ -distributions. The result is

$$\Phi_6 = \langle 45 \rangle^{-1-2\ell_2-\varepsilon_4} \int \frac{dc_{16}dc_{26}dc_{36}}{c_{36}^{1+2\ell_2-2\ell_3-\varepsilon_5}} \phi_6, \tag{3.12}$$

$$\begin{aligned}\phi_6^{-1} &= (\langle 15 \rangle - c_{16}\langle 65 \rangle)^{1-\varepsilon_4} (\langle 12 \rangle - c_{16}\langle 62 \rangle + c_{26}\langle 61 \rangle)^{1-\varepsilon_5} \\ &\quad \times (\langle 12 \rangle c_{36} + \langle 23 \rangle c_{16} + \langle 31 \rangle c_{26})^{-1-2\ell_1+\varepsilon_4+\varepsilon_5} (\langle 24 \rangle c_{36} - \langle 34 \rangle c_{26})^{1+2\ell_1-2\ell_2-\varepsilon_4}.\end{aligned}$$

4 Correlators with deformed symmetry

The symmetry condition for correlators can be extended to the cases with the monodromy matrix operator built from L matrices with algebraic deformations. In applications algebraic deformation may describe particular ways of symmetry breaking.

In the case of trigonometric (q -) deformation and also for the case of elliptic deformation we know the expressions for the L matrices, the YB operators and of its factors in analogy [26]. We discuss the case of quantum deformation only.

The straightforward q -deformations of (2.3) are

$$L_q^+(u) = \begin{pmatrix} [u+1+x_1\partial_1]_q & \frac{x_2}{x_1}[x_1\partial_1]_q \\ \frac{x_1}{x_2}[x_2\partial_2]_q & [u+1+x_2\partial_2]_q \end{pmatrix}, \quad L_q^-(u) = \begin{pmatrix} [u-x_1\partial_1]_q & -\frac{x_1}{x_2}[x_2\partial_2]_q \\ -\frac{x_2}{x_1}[x_1\partial_1]_q & [u-x_2\partial_2]_q \end{pmatrix}. \quad (4.1)$$

We use the notation $[x]_q = (q^x - q^{-x})(q - q^{-1})^{-1}$. We have the factorized forms

$$L_q^+(u) = (q - q^{-1})^{-1} \begin{pmatrix} 1 & 1 \\ -\frac{x_1}{x_2}q^{-u-1-x_1\partial_1-x_2\partial_2} & -\frac{x_1}{x_2}q^{u+1+x_1\partial_1+x_2\partial_2} \end{pmatrix} \begin{pmatrix} q^{x_1\partial_1+1} & 0 \\ 0 & q^{-x_1\partial_1-1} \end{pmatrix} \\ \times \begin{pmatrix} q^u & \frac{x_2}{x_1} \\ -q^{-u} & -\frac{x_2}{x_1} \end{pmatrix},$$

$$L_q^-(u) = (q - q^{-1})^{-1} \begin{pmatrix} 1 & 1 \\ \frac{x_2}{x_1}q^{-u} & \frac{x_2}{x_1}q^u \end{pmatrix} \begin{pmatrix} q^{x_2\partial_2+1} & 0 \\ 0 & q^{-x_2\partial_2-1} \end{pmatrix} \begin{pmatrix} q^{u-1-x_1\partial_1-x_2\partial_2} & -\frac{x_1}{x_2} \\ -q^{-u+1+x_1\partial_1+x_2\partial_2} & \frac{x_1}{x_2} \end{pmatrix}.$$

With these forms it is easy to prove that the relation of inversion (2.6) hold with a modification.

$$\left(\frac{L_q^+(u)}{[u]_q} \right)^{-1} = \frac{L_{q^{-1}}^+(-u-1-(x_1\partial_1+x_2\partial_2))}{[-u-1-(x_1\partial_1+x_2\partial_2)]_q}, \quad (4.2)$$

$$\left(\frac{L_q^-(u)}{[u]_q} \right)^{-1} = \frac{L_{q^{-1}}^-(-u+1+(x_1\partial_1+x_2\partial_2))}{[-u+1+(x_1\partial_1+x_2\partial_2)]_q}.$$

Here x_1, x_2 are the components of a 2-dimensional vector \mathbf{x} . In connection with correlators and YB relation we deal with more than one 2-dimensional vectors and we use the index notation $x_{i,1}, x_{i,2}$ for the components of \mathbf{x}_i . As in the undeformed case we introduce the ratio variables for the point i if the corresponding L matrix is chosen with signature $+$ as $x_i = \frac{x_{i,1}}{x_{i,2}}$ or with signature $-$ as $x_i = -\frac{x_{i,2}}{x_{i,1}}$. Then we have for the restriction to the weight 2ℓ in ratio coordinates

$$L_q^+(u^+, u) \rightarrow L_q(u^+ + 1, u) = \begin{pmatrix} [x\partial + u + 1]_q & x^{-1}[x\partial]_q \\ x[-x\partial + u^+ - u]_q & [u^+ + 1 - x\partial]_q \end{pmatrix} \\ = [u^+ + 1]_q \hat{V}_q^{-1}(u^+ + 1) \hat{D}_q \hat{V}_q(u), \quad (4.3)$$

$$\hat{V}_q(u) = \begin{pmatrix} q^u & x^{-1} \\ -q^{-u} & -x^{-1} \end{pmatrix}, \quad \hat{D}_q = \begin{pmatrix} q^{x\partial+1} & 0 \\ 0 & q^{-x\partial-1} \end{pmatrix},$$

and similar for $L^-(u, u^-)$ in complete analogy to the undeformed case (2.13).

We introduce also $v^{(1)}, v^{(2)}$ as above (2.12) and write the inversion relation in analogy,

$$L_q^{-1}(v^{(1)}, v^{(2)}) = -\frac{1}{[v^{(1)}]_q [v^{(2)}]_q} L_{q^{-1}}(-v^{(2)}, -v^{(1)}). \quad (4.4)$$

The YB operator factors corresponding to the elementary permutations of the parameters $v_1^{(1)}, v_1^{(2)}, v_2^{(1)}, v_2^{(2)}$ in the product of L operators are known [26].

$$\mathcal{S}_{q12}(w) = x_1^w \frac{(\frac{x_2}{x_1} q^{1-w}; q^2)}{(\frac{x_2}{x_1} q^{1+w}; q^2)} = \pi(x_1, x_2; w), \quad w = v_1^{(2)} - v_2^{(1)}. \quad (4.5)$$

The latter notation emphasizes that this is the appropriate deformation of $(x_1 - x_2)^w$. We use the standard notation for the infinite product $(x; q) := \prod_{j=1}^{\infty} (1 - xq^{j-1})$.

The other YB operators of elementary permutations are in analogy given in terms of the intertwiner W_q , $\mathcal{S}_{q11}(v_1^{(1)} - v_1^{(2)}) = W_{q,1}(v_1^{(1)} - v_1^{(2)})$, $\mathcal{S}_{q22}(v_2^{(1)} - v_2^{(2)}) = W_{q,2}(v_2^{(1)} - v_2^{(2)})$.

$$W_{q,i}(a) = \frac{1}{x_i^a} \frac{(q^{2x_i \partial_i + 1 - a}; q^2)}{(q^{2x_i \partial_i + 2}; q^2)} q^{-ax_i \partial_i} = \frac{1}{x_i^a} \frac{\Gamma_q(x_i \partial_i + 1)}{\Gamma_q(x_i \partial_i + 1 - a)}. \quad (4.6)$$

The latter notation emphasizes that the expression can be considered as the deformation of (2.17) with the ratio of Gamma functions. The analogy is useful indeed because there is a q -deformed analogy of the Beta integral [29] allowing to express the action of W_q on functions in analogy to (2.18) as

$$W_q(a)f(x) = \text{const} \int d\tilde{x} \pi(x, \tilde{x}; -a - 1) f(\tilde{x}). \quad (4.7)$$

The YB relation for \mathcal{S}_{q12} can be checked directly using the factorized form (4.3) [26], see Appendix C.

$$\mathcal{S}_{q12}(v_1^{(2)} - v_2^{(1)}) L_1(v_1^{(1)}, v_1^{(2)}) L_2(v_2^{(1)}, v_2^{(2)}) = L_1(v_1^{(1)}, v_2^{(1)}) L_2(v_1^{(2)}, v_2^{(2)}) \mathcal{S}_{q12}(v_1^{(2)} - v_2^{(1)}). \quad (4.8)$$

From the deformed $\mathcal{S}_{q12}(u)$ it is straightforward to reconstruct the deformed R_{q12}^{+-} ,

$$R_{q12}^{+-}(w) = (x_{1,1} x_{2,1})^w \cdot \frac{(\mathcal{X} q^{1-w}; q^2)}{(\mathcal{X} q^{1+w}; q^2)}, \quad \mathcal{X} = -\frac{x_{1,2} x_{2,2}}{x_{1,1} x_{2,1}}. \quad (4.9)$$

The factorized form in homogeneous coordinates allows to repeat the above calculation and to check the YB relation (see Appendix C)

$$R_{q12}^{+-}(u_1 - u_2) L_{q1}^+(u_1) L_{q2}^-(u_2) = L_{q1}^+(u_2) L_{q2}^-(u_1) R_{q12}(u_1 - u_2). \quad (4.10)$$

The examples of symmetric correlator constructions working in ratio coordinates and the YB operators of elementary permutations work in the deformed case by straightforward analogy. The basic correlator 1 from which we generate the result by R operations has vanishing weights. The resulting correlator has weights calculated from the permutation of parameters to $v_1^{(1)}, v_1^{(2)}; v_2^{(1)}, v_2^{(2)}; v_3^{(1)}, v_3^{(2)}; \dots$ as $2\ell_i + 1 = v_i^{(1)} - v_i^{(2)}$.

The action of (4.8) on 1 shows that

$$\mathcal{S}_{q12}(w) = \pi(x_1, x_2; w) = x_1^w \frac{(\frac{x_2}{x_1} q^{1-w}; q^2)}{(\frac{x_2}{x_1} q^{1+w}; q^2)}, \quad T_2 : v_1, v_2; v_1 - 1, v_2 - 1,$$

with $w = v_1 - v_2 - 1$ is the symmetric two-point correlator for the monodromy T_2 .

Let us mention an example of a 3-point correlator. We have

$$\begin{aligned} \mathcal{S}_{q23}(v_2^{(2)} - v_3^{(1)}) \mathcal{S}_{q12}(v_1^{(2)} - v_2^{(1)}) L_1(v_1^{(1)}, v_1^{(2)}) L_2(v_2^{(1)}, v_2^{(2)}) L_3(v_3^{(1)}, v_3^{(2)}) \\ = L_1(v_1^{(1)}, v_2^{(1)}) L_2(v_1^{(2)}, v_3^{(1)}) L_3(v_2^{(2)}, v_3^{(2)}) \mathcal{S}_{q23}(v_2^{(2)} - v_3^{(1)}) \mathcal{S}_{q12}(v_1^{(2)} - v_2^{(1)}). \end{aligned}$$

Applying both sides on the constant function 1 we obtain the symmetric 3-point correlator for the monodromy $T_3 : v_1, v_2; v_1 - 1, v_3; v_2 - 1, v_3 - 1$ as

$$\mathcal{S}_{q23}(v_2^{(2)} - v_3^{(1)})\mathcal{S}_{q12}(v_1^{(2)} - v_2^{(1)}) = x_1^{v_1-v_2-1}x_2^{v_2-v_3-1} \frac{(\frac{x_3}{x_2}q^{2+v_3-v_2}; q^2)}{(\frac{x_3}{x_2}q^{v_2-v_3}; q^2)} \frac{(\frac{x_2}{x_1}q^{2+v_2-v_1}; q^2)}{(\frac{x_2}{x_1}q^{v_1-v_2}; q^2)},$$

The eigenvalue is $E(v) = [v_1]_q[v_2]_q[v_3]_q$.

The result of the 3-point correlator (3.1) and the ones of the two steps of its construction lead to the corresponding deformed correlators with the monodromy characterized by the same parameter strings with the substitutions

$$(x_i - x_j)^w \rightarrow \pi(x_i, x_j; w).$$

This substitution rule works for all YSC which can be generated by \mathcal{S}_{ij} actions ($i \neq j$) in this way, in particular for the 4-point YSC (3.10).

5 Integral operators from symmetric correlators

We consider integral operators with YSC as kernels. The symmetry of the correlators expressed in the monodromy relation (2.2) implies the symmetry of the operators by the relation of inversion and transposition obeyed by the involved L operators.

5.1 Generalized Yang-Baxter operators

The YB operators considered above map the tensor product of representation spaces as $V_{\ell_1} \otimes V_{\ell_2} \rightarrow V_{\tilde{\ell}_1} \otimes V_{\tilde{\ell}_2}$ and obey a relation with a product of two L matrices of the form

$$R L_1^+(u_1^+, u_1) L_2^+(u_2^+, u_2) = L_{\sigma(1)}^+(u_{\sigma(1)}^+, u'_{\sigma(1)}) L_{\sigma(2)}^+(u_{\sigma(2)}^+, u'_{\sigma(2)}) R.$$

We consider generalized R operators mapping tensor products with more factors and obeying the more general relation involving monodromy matrices of M factors.

$$R L_1^+(u_1^+, u_1) \cdots L_M^+(u_M^+, u_M) = L_{\sigma(1)}^+(u_{\sigma(1)}^+, u'_{\sigma(1)}) \cdots L_{\sigma(M)}^+(u_{\sigma(M)}^+, u'_{\sigma(M)}) R. \quad (5.1)$$

The interchange results in another monodromy matrix with the L factors permuted, $1, \dots, M \rightarrow \sigma(1), \dots, \sigma(M)$ and the sets of parameters substituted $u_1^+, \dots, u_M^+ \rightarrow u_{\sigma(1)}^+, \dots, u_{\sigma(M)}^+$, and $u_1, \dots, u_M \rightarrow u'_1, \dots, u'_M$.

Examples of such operators are obtained by constructing integral operators with the kernels chosen as particular Yangian symmetric correlators. The Yangian symmetry condition then implies a generalized YB relation. Indeed, let the operator acting on a function of M points be defined as

$$R\psi(\mathbf{x}_1, \dots, \mathbf{x}_M) = \int d\mathbf{x}'_1 \cdots d\mathbf{x}'_M \psi(\mathbf{x}'_1, \dots, \mathbf{x}'_M) \Phi(\mathbf{x}'_1, \dots, \mathbf{x}'_M, \mathbf{x}_1, \dots, \mathbf{x}_M). \quad (5.2)$$

Both the function and the kernel depend homogeneously on the coordinates of the points with weights $2\tilde{\ell}_1, \dots, 2\tilde{\ell}_M$ for the function and $2\ell'_1, \dots, 2\ell'_M, 2\ell_1, \dots, 2\ell_M$ for the kernel with the relations

$$2\ell'_i + 2\tilde{\ell}_i + 2 = 0, \quad i = 1, \dots, M.$$

The resulting function has the weights $2\ell_1, \dots, 2\ell_M$, and the case $2\ell_i = 2\tilde{\ell}_i$ will be considered in particular.

The Yangian symmetry condition obeyed by the kernel Φ has the general form (2.2) and can be rewritten after relabeling $1, \dots, N \rightarrow 1', \dots, M', 1, \dots, M$ as

$$L_1^+(u_1^+, u_1) \cdots L_M^+(u_M^+, u_M) \Phi = E(\mathbf{u}) L_{M'}^{+-1}(u_{M'}^+, u_{M'}) \cdots L_{1'}^{+-1}(u_{1'}^+, u_{1'}) \Phi.$$

We consider the action of $L_1^+(u_1^+, u_1) \cdots L_M^+(u_M^+, u_M)$ onto $R\psi$, apply the latter relation for the kernel and move by transposition the action of $L_{M'}^{+-1}(u_{M'}^+, u_{M'}) \cdots L_{1'}^{+-1}(u_{1'}^+, u_{1'})$ from the kernel to the function.

$$\begin{aligned} & L_1^+(u_1^+, u_1) \cdots L_M^+(u_M^+, u_M) R\psi(\mathbf{x}_1, \dots, \mathbf{x}_M) \\ &= \int d\mathbf{x}'_1 \cdots d\mathbf{x}'_M \psi(\mathbf{x}'_1, \dots, \mathbf{x}'_M) \cdot L_1^+(u_1^+, u_1) \cdots L_M^+(u_M^+, u_M) \Phi(\mathbf{x}'_1, \dots, \mathbf{x}'_M, \mathbf{x}_1, \dots, \mathbf{x}_M) \\ &= E(\mathbf{u}) \int d\mathbf{x}'_1 \cdots d\mathbf{x}'_M \left[L_{M'}^{+-1T}(u_{M'}^+, u_{M'}) \cdots L_{1'}^{+-1T}(u_{1'}^+, u_{1'}) \psi(\mathbf{x}'_1, \dots, \mathbf{x}'_M) \right] \\ & \quad \times \Phi(\mathbf{x}'_1, \dots, \mathbf{x}'_M, \mathbf{x}_1, \dots, \mathbf{x}_M). \end{aligned}$$

We have the relation for inversion and operator conjugation of the L matrices

$$(L^+(u^+, u))^{-1T} = \frac{1}{u(u^+ + 1)} L^+(u - 2, u^+). \quad (5.3)$$

We abbreviate

$$F(\mathbf{u}) = \frac{E(\mathbf{u})}{u_{1'}(u_{1'}^+ + 1) \cdots u_{M'}(u_{M'}^+ + 1)}.$$

Then we obtain

$$L_1^+(u_1^+, u_1) \cdots L_M^+(u_M^+, u_M) R\psi = F(\mathbf{u}) R L_{M'}^+(u_{M'} - 2, u_{M'}^+) \cdots L_{1'}^+(u_{1'} - 2, u_{1'}^+) \psi.$$

The result is of the form of (5.1) with $\sigma(1, 2, \dots, M) = M, M - 1, \dots, 1$ and $u_{i'} \rightarrow u_{i'}^+$, $u_{i'}^+ \rightarrow u_{i'} - 2$.

Integral operators with YSC kernels act in a highly symmetric way, defining homomorphisms not only of the sl_2 Lie algebra action on the functions (global symmetry) but of the related Yangian algebra. The resulting YB relations encode this symmetry property. It is also reflected in the eigenvalue spectrum.

5.2 Examples

We specify the construction of Yangian symmetric operators from YSC in the cases $M = 2$ and $M = 3$. In the first case we obtain operators with simple commutation relations with a monodromy out of two L operators, in particular the standard RLL Yang-Baxter relation. We use as kernels the YSC from the stepwise construction (3.6, 3.7). In the case $M = 3$ we use the 6-point YSC (3.12) and the related monodromy (3.11) to obtain the generalized YB relation involving the monodromy out of three L operators.

Symmetric two-point operators

We consider first integral operators with 4-point YSC as kernels obeying

$$T_{1,2,3,4}(u) \Phi^{--++} = E(u) \Phi^{--++} \quad (5.4)$$

with the monodromy

$$T_{1,2,3,4}(u) = L_1^+(u_1^+, u_1) L_2^+(u_2^+, u_2) L_3^+(u_3^+, u_3) L_4^+(u_4^+, u_4)$$

resulting from the original monodromy $L_1^+(u_1^{0+}, u_1^0) \cdots L_4^+(u_4^{0+}, u_4^0)$ if the correlator is generated by R^{++} operations from the basic correlator $\delta^{(2)}(\mathbf{x}_1)\delta^{(2)}(\mathbf{x}_2)$. We denote those correlators by superscript $--++$. Parameters u_1, \dots, u_4 and u_1^+, \dots, u_4^+ are permutations of u_1^0, \dots, u_4^0 and $u_1^{0+} = u_1^0 - 2, u_2^{0+} = u_2^0 - 2, u_3^{0+} = u_3^0, u_4^{0+} = u_4^0$. The eigenvalue function is $E(u) = u_1^0 u_2^0 (u_3^0 + 1)(u_4^0 + 1)$.

We consider now functions $\psi(\mathbf{x}_1, \mathbf{x}_2)$ with the weights $2\tilde{\ell}_1$ and $2\tilde{\ell}_2$ at the two points and the operator defined by the specification of (5.1)

$$\hat{Q}\psi(\mathbf{x}_1, \mathbf{x}_2) = \int d\mathbf{x}_1' d\mathbf{x}_2' \psi(\mathbf{x}_1', \mathbf{x}_2') \Phi^{--++}(\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_1, \mathbf{x}_2). \quad (5.5)$$

Recall that at this step the initial labels of the points have been changed as $1, 2, 3, 4 \rightarrow 1', 2', 1, 2$. The scale symmetry of the integration requires for the weights at the corresponding correlator points

$$2\ell_{i'} = -2 - 2\tilde{\ell}_i.$$

Consider the action of $L_1^+(u_1^+, u_1)L_2^+(u_2^+, u_2)$ on $\hat{Q}\psi(\mathbf{x}_1, \mathbf{x}_2)$ and rewrite the symmetry condition (5.4) in the redefined labels as

$$L_1^+(u_1^+, u_1)L_2^+(u_2^+, u_2)\Phi^{--++}(x_1', x_2', x_1, x_2) = E(u)L_{2'}^{+-1}(u_2^+, u_2)L_{1'}^{+-1}(u_1^+, u_1)\Phi^{--++}$$

and conclude as above

$$\hat{Q}L_2^+(u_2' - 2, u_2^+)L_1^+(u_1' - 2, u_1^+)F(u) = L_1^+(u_1^+, u_1)L_2^+(u_2^+, u_2)\hat{Q}, \quad (5.6)$$

$$F(u) = \frac{u_1^0 u_2^0 (u_1^0 + 1)(u_2^0 + 1)}{u_1' (u_1^+ + 1) u_2' (u_2^+ + 1)}.$$

The cases of monodromies are specified by the permutation pattern

$$\begin{pmatrix} \sigma(1') & \sigma(2') & \sigma(1) & \sigma(2) \\ \bar{\sigma}(1') & \bar{\sigma}(2') & \bar{\sigma}(1) & \bar{\sigma}(2) \end{pmatrix}$$

meaning in particular the relation between the original parameters u_i^0, u_i^{0+} and the ones appearing in (5.4), (5.6), $u_i = u_{\sigma(i)}^0$, $u_i^+ = u_{\bar{\sigma}(i)}^{0+}$. In the following we shall formulate this as substitution rules for the parameters in (5.6) in terms of the original ones and omit the superscript 0 after substitution.

Point permutation operator

In the case of Δ^{--++} (3.6) we have

$$\begin{pmatrix} 1' & 2' & 1 & 2 \\ 2 & 1 & 2' & 1' \end{pmatrix}.$$

This means

$$\begin{aligned} u_{1'} &\rightarrow u_{1'}, u_{2'} \rightarrow u_{2'}, u_1 \rightarrow u_1, u_2 \rightarrow u_2 \\ u_{1'}^+ &\rightarrow u_2, u_{2'}^+ \rightarrow u_1, u_1^+ \rightarrow u_{2'} - 2, u_2^+ \rightarrow u_{1'} - 2 \end{aligned}$$

and implies $F(u) = 1$. We obtain as the specification of (5.6) that the operator $\hat{\Delta}$ defined with this kernel obeys

$$\hat{\Delta}L_2^+(u_{2'} - 2, u_1)L_1^+(u_{1'} - 2, u_2) = L_1^+(u_{2'} - 2, u_1)L_2^+(u_{1'} - 2, u_2)\hat{\Delta}.$$

Thus choosing Δ^{--++} as the kernel in (5.5) one represents the operator of permutation P_{12} acting as

$$P_{12}(\mathbf{x}_1, \mathbf{p}_1)P_{12} = (\mathbf{x}_2, \mathbf{p}_2).$$

Parameter pair permutation operators

The kernel $R_{21}^{++}\Delta^{--++}$ defines by (5.5) the operator $\hat{R}_{12}^{++} = P_{12}R_{12}^{++}$. Indeed, the permutation pattern after relabeling is

$$\begin{pmatrix} 1' & 2' & 1 & 2 \\ 1 & 2 & 2' & 1' \end{pmatrix}.$$

This means

$$\begin{aligned} u_{1'} &\rightarrow u_{1'}, u_{2'} \rightarrow u_{2'}, u_1 \rightarrow u_1, u_2 \rightarrow u_2 \\ u_{1'}^+ &\rightarrow u_1, u_{2'}^+ \rightarrow u_2, u_1^+ \rightarrow u_{2'} - 2, u_2^+ \rightarrow u_{1'} - 2. \end{aligned}$$

We see that $F(u) = 1$ and the relation (5.6) reads

$$\hat{R}_{12}^{++}L_2^+(u_{2'} - 2, u_2)L_1^+(u_{1'} - 2, u_1) = L_1^+(u_{2'} - 2, u_1)L_2^+(u_{1'} - 2, u_2)\hat{R}_{12}^{++}.$$

The parameters u_1, u_2 and the underlying canonical pairs are permuted.

The kernel $R_{12}^{++}\Delta^{--++}$ defines by (5.5) the operator $\hat{R}_{21}^{++} = P_{12}R_{21}^{++}$. Indeed, the permutation pattern after relabeling is

$$\begin{pmatrix} 2' & 1' & 1 & 2 \\ 2 & 1 & 2' & 1' \end{pmatrix}.$$

This means

$$\begin{aligned} u_{1'} &\rightarrow u_{2'}, u_{2'} \rightarrow u_{1'}, u_1 \rightarrow u_1, u_2 \rightarrow u_2, \\ u_{1'}^+ &\rightarrow u_2, u_{2'}^+ \rightarrow u_1, u_1^+ \rightarrow u_{2'} - 2, u_2^+ \rightarrow u_{1'} - 2. \end{aligned}$$

Again $F(u) = 1$. The relation (5.6) results in

$$\hat{R}_{21}^{++}L_2(u_{1'} - 2, u_1)L_1(u_{2'} - 2, u_2) = L_1(u_{2'} - 2, u_1)L_2(u_{1'} - 2, u_2)\hat{R}_{21}^{++}.$$

The parameters $\tilde{u}_1^+ = u_{2'} - 2, \tilde{u}_2^+ = u_{1'} - 2$ are permuted together with the underlying canonical pairs.

The complete Yang-Baxter operator

The kernel $R_{12}^{++}R_{21}^{++}\Delta^{--++} = \Phi^{--++}$ (3.7) defines by (5.5) the complete YB operator $\hat{\mathbb{R}}_{12}$. Indeed, the permutation pattern after relabeling is

$$\begin{pmatrix} 2' & 1' & 1 & 2 \\ 1 & 2 & 2' & 1' \end{pmatrix}.$$

This means

$$\begin{aligned} u_{1'} &\rightarrow u_{2'}, u_{2'} \rightarrow u_{1'}, u_1 \rightarrow u_1, u_2 \rightarrow u_2, \\ u_{1'}^+ &\rightarrow u_1, u_{2'}^+ \rightarrow u_2, u_1^+ \rightarrow u_{2'} - 2, u_2^+ \rightarrow u_{1'} - 2. \end{aligned}$$

We find $F(u) = 1$ and the relation (5.6) results in

$$\hat{\mathbb{R}}_{12}(u_2 - u_1)L_2^+(u_{1'} - 2, u_2)L_1^+(u_{2'} - 2, u_1) = L_1^+(u_{2'} - 2, u_1)L_2^+(u_{1'} - 2, u_2)\hat{\mathbb{R}}_{12}(u_2 - u_1). \quad (5.7)$$

Both parameters and the underlying canonical pairs are permuted. The weights are $2\ell_1 = u_{2'} - u_1 - 2$, $2\ell_2 = u_{1'} - u_2 - 2$ and after fixing their values one can omit the u^+ arguments in $L_i^+(u_i^+, u_i)$ to match the conventional form. The explicit form of the kernel is given by (3.7) with the substitution $1, 2, 3, 4 \rightarrow 1', 2', 1, 2$.

The completely connected 4-point correlator has been constructed by other R sequences, e.g. string with the correlator (3.5). Corresponding to the permutation pattern

$$\begin{pmatrix} 1' & 2' & 1 & 2 \\ 1 & 2 & 1' & 2' \end{pmatrix}$$

one obtains again the normal YB relation (5.7) with other parameter substitutions and with the weights calculated differently from the original spectral parameters.

Note that the permutation P_{12} can be removed by modifying the definition (5.5) of the integral operator by a permutation of $1, 2$ or $1', 2'$ in the step of relabeling the original $1, 2, 3, 4$.

Explicit kernel in ratio coordinates

The YB relation with $\hat{\mathbb{R}}_{12}(u_1 - u_2)$ (5.7) can be written in the ratio coordinate form (sect. 2.3) with $\hat{R}_{12}(v_1 - v_2) = P_{12}R_{12}^1R_{12}^2$

$$\hat{R}_{12}(v_1 - v_2)L_1(v_1^{(1)}, v_1^{(2)})L_2(v_2^{(1)}, v_2^{(2)}) = L_2(v_2^{(1)}, v_2^{(2)})L_1(v_1^{(1)}, v_1^{(2)})\hat{R}_{12}(v_1 - v_2). \quad (5.8)$$

The relation of reduction of $L^+(u)$ to $L(v^{(1)}, v^{(2)})$ (2.11) implies that the factor in ratio coordinates in the 4-point correlator (3.7)

$$\frac{x_{12}^{1+2\ell_1}}{x_{14}x_{23}^{1+2\ell_1-2\ell_2}x_{34}^{1+2\ell_2}} \left(\frac{x_{12}x_{34}}{x_{14}x_{23}} \right)^{u_4-u_3}$$

results in the kernel of $P_{12}R_{12}$. Our convention about the notations u and v is $v_i = u_i + \ell_i$, i.e. $u_4 - u_3 = v_4 - v_3 + \ell_3 - \ell_4$. We recall that the weights of the correlator (3.7) are related as $2\ell_1 + 2\ell_3 + 2 = 0$, $2\ell_2 + 2\ell_4 + 2 = 0$. According to (5.5) the kernel of $P_{12}R_{12}$ is obtained from the resulting expression by relabeling the points as $1, 2, 3, 4 \rightarrow 1', 2', 1, 2$. Finally, we obtain the kernel of $R_{12}(v)$, $v = v_2 - v_1$, (without the permutation P_{12}) by permuting $1', 2'$.

$$\tilde{\Phi}^{--++} = \frac{x_{12}^{1+\ell_1+\ell_2}}{x_{11'}^{1+\ell_2-\ell_1}x_{22'}^{1+\ell_1-\ell_2}x_{1'2'}^{1+\ell_1+\ell_2}} \left(\frac{x_{12}x_{1'2'}}{x_{11'}x_{22'}} \right)^v. \quad (5.9)$$

This result can be derived directly from the RLL relation (5.8) using the factorized form of $L(v^{(1)}, v^{(2)})$ (2.13), see e.g. [19].

A three-point generalized Yang-Baxter operator

We have constructed the 6-point correlator Φ_6 (3.12) with the weight relations with the monodromy (3.11). The correlator has been derived from the condition

$$T_6\Phi_6 = E(u)\Phi_6, \quad (5.10)$$

$$E(u) = u_1u_2u_3(u_4+1)(u_5+1)(u_6+1).$$

We relabel the points as $1, 2, 3, 4, 5, 6 \rightarrow 1', 2', 3', 1, 2, 3$ and consider the integral operator acting as

$$\hat{Q}\psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \int d\mathbf{x}'_1 d\mathbf{x}'_2 d\mathbf{x}'_3 \psi(\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3) \Phi_6(\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3). \quad (5.11)$$

Following the general scheme we act on $\hat{Q}\psi$ (in the new labels) by $L_1^+(\varepsilon_4 + \varepsilon_5 + u)L_2^+(\varepsilon_5 + u)L_3^+(u)$. We use the monodromy eigenvalue relation for Φ_6 (5.10) rewritten by applying the inversion and transposition relation (5.3). $E(u)$ cancels and we obtain that $\hat{Q} = \mathbb{R}_{321}$ obeys the following generalized YB relation (we omit the arguments $u_i^+ = u_i + 2\ell_i$, $i = 1, 2, 3$)

$$\hat{\mathbb{R}}_{321}(\varepsilon_4, \varepsilon_5)L_3^+(u)L_2^+(\varepsilon_5 + u)L_1^+(\varepsilon_4 + \varepsilon_5 + u) = L_1^+(\varepsilon_4 + \varepsilon_5 + u)L_2^+(\varepsilon_5 + u)L_3^+(u)\hat{\mathbb{R}}_{321}(\varepsilon_4, \varepsilon_5). \quad (5.12)$$

5.3 Kernels of QCD parton evolution

The generalized parton distributions [17] contribute to hard-exclusive production processes. The form in (one-dimensional, light-cone component) momenta is normally used. The form in positions (on the light ray) is convenient for representing the conformal symmetry actions and is useful also in the related operator product renormalization. For explaining this schematically (for more details cf. [28]) let us consider composite operators of the form

$$D_{\mu_1} \dots D_{\mu_n} \psi(0) \cdot D_{\nu_1} \dots D_{\nu_m} \phi(0)$$

By contracting the indices with parallel light-like vectors $x_1^\mu, x_2^\nu, x_{1,\mu}x_1^\mu = x_{2,\mu}x_2^\mu = x_{1,\mu}x_2^\mu = 0$ one projects on the symmetric traceless part. Let us choose the axial gauge $x_1^\mu A_\mu = 0$, then $x^\mu D_\mu = x\partial$. It is natural to consider the generating function of such operator product projections

$$\sum_{n,m} \frac{x_1^n}{n!} \partial^n \psi(0) \cdot \frac{x_2^m}{m!} \partial^m \phi(0) = \psi(x_1) \phi(x_2)$$

The resulting operator valued function of two points on the light ray can be substituted instead of $\psi(x_1, x_2)$ in the definition of the integral operator (5.5). Operator products involving more than two fields can be treated analogously. Then we deal with functions of M points on the light ray and integral operators with $2M$ point YSC as kernels. Choosing the kernel appropriately, this results in a convenient formulation of the renormalization of a class of composite operators.

In QCD, if choosing the light-cone components of the quark and gluon field, these composite operators are called quasi-partonic [27]. In $\mathcal{N} = 4$ super Yang Mills theory one has analogous composite operators with components of the superfield. With combinations of the scalar components one has in particular the $SL(2)$ sector [20] such that the mixing with the products involving other components is absent. The spin chain treatment of the latter has attracted much attention because of the non-compact representations at the sites.

The ordinary parton distributions [5] contribute to the (inclusive) deep-inelastic scattering structure function. Their evolution kernels correspond to the vanishing momentum transfer limit $k_1 + k_2 = 0$ of the kernels of generalized parton distributions.

We consider the kernel of the YB operator $R_{12}(v)$ in ratio coordinate form (5.9). For real values of the positions and if $x_{1'}, x_{2'}$ lie in between x_1, x_2 we can write this in the simplex integral form. We consider the case $\ell_1 = \ell_2 = \ell$.

$$\begin{aligned} \tilde{\Phi}^{---++} &= \int_0^1 d^3\alpha \delta(\alpha_1 + \alpha_2 + \alpha_3 - 1) \alpha_1^{-1-v} \alpha_2^{-1-v} \alpha_3^{-1+v-2\ell} \delta(x_{11'} - \alpha_1 x_{12}) \delta(x_{22'} + \alpha_2 x_{12}) \\ &\equiv J_{-v, -v, v-2\ell}. \end{aligned}$$

This form is the convenient one for comparison with the evolution kernels of generalized parton distribution as given in [21]. It is also convenient because the Fourier transformation is easily done. The resulting form is the one of [22] and J_{n_1, n_2, n_3} is the notation for the terms appearing in the kernels introduced in that papers.

In the limit $v \rightarrow 0$ and in the negative integer values for $v = -1, -2$ we find the particular cases related to the parton kernels [21].

At $v \rightarrow 0$ we find in the residues at $\alpha_1 = 0$ and $\alpha_2 = 0$ the kernel of the unit operator $\delta(x_{11'})\delta(x_{22'})$ and also

$$J_{11'}^{(-2h)} + J_{22'}^{(-2h)}, \quad J_{11'}^{(-2h)} = \int d\alpha \frac{(1-\alpha)^{-2h}}{[\alpha]_+} \delta(x_{11'}) \delta(x_{22'} + \alpha x_{12}).$$

At $v = -1$ we have from (5.9) the kernel

$$J_{1,1,-1-2\ell} = \left(\frac{x_{1'2'}}{x_{12}} \right)^{-2-2\ell} x_{12}^{-2}.$$

At $v = -2$ we have (5.9)

$$J_{2,2,-2-2\ell} = \left(\frac{x_{1'2'}}{x_{12}} \right)^{-3-2\ell} x_{12}^{-4} x_{11'} x_{22'}.$$

We substitute the weight ℓ or the scaling helicity h for quarks as $\ell = -1, h = -\frac{1}{2}$ and for gluons as $\ell = -\frac{3}{2}, h = -1$.

In both the kernels for gluons and quarks of parallel helicities only one value of the extra Yangian representation parameter v contributes, namely $v = 0$.

In the kernel of anti-parallel helicity quarks two values of v contribute, $v = 0, -1$. In the kernel of anti-parallel helicity gluons three values of v contribute, $v = 0, -1, -2$.

In the case of quarks $h = -\frac{1}{2}$ the contribution at $v = -2$ is singular. Physically such a contribution occurs in the flavor singlet channel only and it is proportional to $\delta(x_{1'} - x_{2'})$. We note that the comparison also works in the case of gluon-quark interaction kernels with $h_1 = -1, h_2 = -\frac{1}{2}$ and $v \rightarrow \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.

The leading order renormalization in the $SL(2)$ sector of $\mathcal{N} = 4$ super Yang Mills corresponds to the kernel (5.9) at $h_1 = h_2 = 0$ and $v \rightarrow 0$.

In the case of ordinary parton distribution kernels where the momentum transfer vanishes we have the result (3.9) of the symmetric correlator. In order to obtain the kernels in one-dimensional momenta we have to remove the factor carrying the scaling weights and to do the same substitution of parameters as leading to $R_{12}(v)$ in positions. First we have to do the substitution (3.8),

$$\varepsilon = u_2 - u_3 \rightarrow u_4 - u_3 - 2\ell_2 = v_4 - v_3 + \ell_3 + \ell_4 + 2.$$

In the last step we have used our convention about the relation between the notations u and v , $u_i = v_i + \ell_i$, and the relation of the weights in the considered 4-point correlator, $2\ell_2 + 2\ell_4 + 2 = 0$. Finally we change to helicities according to $2\ell_i + 1 = 2h_i$ to obtain the substitution rule of $\varepsilon = u_2 - u_3$ by $v = v_4 - v_3$ after relabeling the points $1234 \rightarrow 1'2'12$ as

$$\varepsilon \rightarrow v + h_1 + h_2 + 1.$$

This results in

$$(k_1 k_{1'})^{-h_1 - h_2 + v} (k_{1'} - k_1)^{-1 - 2v}. \quad (5.13)$$

We use the momentum fraction $z = \frac{k_1}{k_{1'}}$ and consider the same cases as above. The case of parallel helicity appears in the extraordinary situations of quark structure functions of odd chirality and of the photon structure function F_3^γ . The kernels are [23, 24] proportional to

$$\frac{z}{(1-z)}, \quad \frac{z^2}{(1-z)}$$

and they result as expected from (5.13) at $v = 0$ and for $h_1 = h_2 = -\frac{1}{2}$ and $h_1 = h_2 = -1$, respectively.

The ordinary parton distributions have a scale dependence governed by kernels composed as sums of contributions from (5.13) at $v = 0, -1, -2$ for gluons and $v = 0, -1$ for quarks. The coefficients in the sums do not follow from the considered symmetry. We find in comparison with the formulation in [27] for gluons ($h_1 = h_2 = -1$)

$$zw_{+1}^{+1} = \frac{1+z^4}{(1-z)} = 2 \cdot \frac{z^2}{(1-z)} + 4 \cdot z(1-z) + (1-z)^3,$$

and for quarks ($h_1 = h_2 = -\frac{1}{2}$)

$$w_{+\frac{1}{2}}^{+\frac{1}{2}} = \frac{1+z^2}{(1-z)} = 2 \cdot \frac{z}{(1-z)} + (1-z).$$

w_{+1}^{+1} and $w_{+\frac{1}{2}}^{+\frac{1}{2}}$ describe the probability distribution in momentum fractions of the parton splitting of gluons or quarks with no flip of spin. w_{+1}^{-1} describes the gluon splitting with spin flip,

$$zw_{+1}^{-1} = (1-z)^3.$$

It corresponds to our expression (5.13) at $h_1 = h_2 = -1$ and $v = -2$. A quark splitting with flip cannot occur in the flavor non-singlet channel at leading order, $w_{+\frac{1}{2}}^{-\frac{1}{2}} = 0$. This corresponds to our statement that there is no $v = -2$ contribution in the quark case.

The scale evolution of the unpolarized parton distributions are determined by the sum of the splitting kernels with and without spin flip, $w_{+1}^{+1} + w_{+1}^{-1}$, $w_{+\frac{1}{2}}^{+\frac{1}{2}} + w_{+\frac{1}{2}}^{-\frac{1}{2}} = w_{+\frac{1}{2}}^{+\frac{1}{2}}$. The scale evolution of the polarized (helicity asymmetry) parton distributions is determined by the differences $w_{+1}^{+1} - w_{+1}^{-1}$ for gluons and $w_{+\frac{1}{2}}^{+\frac{1}{2}} - w_{+\frac{1}{2}}^{-\frac{1}{2}} = w_{+\frac{1}{2}}^{+\frac{1}{2}}$ for quarks (flavor non-singlet). They correspond to our expression (5.13) at $v = 0, -1$ with the corresponding values of $h_1 = h_2$.

In the expressions written above the regularization at $z = 1, z = 0$ is not explicit. Recall that our definition of the integral operators assumes integration contours which allow integration by parts (denoted as conjugation T) without boundary terms. Here the integration in z over the segment $(0, 1)$ is to be understood as the result of the contraction of a closed contour. This results in the conventional formulation for parton splitting kernels in terms of distributions like $\frac{1}{[1-z]_+}$.

6 Spectra of symmetric integral operators

We consider operators defined in Sect. 5.1 by (5.2). We choose as kernel a correlator with $2M$ points $1', \dots, M', 1, \dots, M$ with the weight balance

$$2\ell_i + 2\ell'_i + 2 = 0, \quad i = 1, \dots, M.$$

This implies

$$2\ell_i = -2 - 2\ell'_i = 2\tilde{\ell}_i.$$

In this way the weights of the resulting function $R\psi$ are the same as the ones of ψ . This ensures the consistency of the eigenvalue problem on functions depending homogeneously on M two-dimensional points with the weights $2\tilde{\ell}_1, \dots, 2\tilde{\ell}_M$,

$$R\psi = \lambda\psi. \quad (6.1)$$

The Yangian symmetry of the operator induced by the symmetry of its kernel has strong implications on the spectrum. In the case of the two-point operator obeying the standard RLL relation the calculation of the spectrum is well known (cf. [30]). We recall it for comparison with the generalized case. We describe the solution of the spectral problem in the case of the symmetric 3-point operator. The outlined scheme can be generalized to more points. For applications it is important to have algorithms for calculating the complete spectrum exactly.

6.1 The spectrum of the YB operator

The integral operator obtained from the kernel Φ^{--++} (3.7) represents a Yang-Baxter operator intertwining the representations $2\ell_1$ and $2\ell_2$ according to (5.7). In this YB relation we change to the ratio coordinate form (5.8) and write the L matrix in terms of sl_2 generators S^a as (2.14). Let $\psi(x_1, x_2)$ obey the eigenvalue relation (the specification of (6.1)),

$$R_{21}\psi(x_1, x_2) = \lambda\psi(x_1, x_2).$$

The YB relation implies

$$R_{21}L_2(v)L_1(v - \delta)\psi(x_1, x_2) = \lambda L_1(v - \delta)L_2(v)\psi(x_1, x_2). \quad (6.2)$$

We abbreviate the relation as $R\mathcal{L}\psi = \lambda\mathcal{R}\psi$ and expand in powers of v . At v^2 we have a trivial relation, at v^1 we find: If ψ is an eigenfunction then it is also $(S_1^a + S_2^a)\psi$ with the same eigenvalue, i.e. we have sl_2 irreducible representation subspaces of degeneracy. The conditions at v^0 are related by this global symmetry. Therefore it is sufficient to analyze the condition related to one matrix element, e.g. 12.

$$\mathcal{R}_{12} = (S_1^0 - \delta)S_2^- - S_1^-S_2^0, \quad \mathcal{L}_{12} = S_2^0S_1^- - S_2^-(S_1^0 + \delta).$$

We consider the function

$$\psi_n^{(0)} = (x_1 - x_2)^n$$

being a lowest weight vector

$$(S_1^- + S_2^-)\psi_n^{(0)} = 0, \quad (S_1^0 + S_2^0)\psi_n^{(0)} = -\mu_n\psi_n^{(0)}, \quad \mu_n = -\ell_1 - \ell_2 + n.$$

We calculate

$$\mathcal{R}_{12}\psi_n^{(0)} = \frac{1}{2}\mu_n S^- \psi_n^{(0)} + \frac{1}{2}(1 + \delta)S^- \psi_n^{(0)}, \quad \mathcal{L}_{12}\psi_n^{(0)} = -\frac{1}{2}\mu_n S^- \psi_n^{(0)} - \frac{1}{2}(1 - \delta)S^- \psi_n^{(0)},$$

where $S^a = S_1^a - S_2^a$. We obtain by (6.2)

$$\lambda_n[\mu_n + 1 + \delta]S^- \psi_n^{(0)} = [-\mu_n + \delta - 1]R S^- \psi_n^{(0)}.$$

We have marked the eigenvalue of the representation generated from the lowest weight vector $\psi_n^{(0)}$ by the index n . We find that also $S^-\psi_n^{(0)}$ is an eigenfunction of R_{21} with another eigenvalue,

$$R_{21} S^-\psi_n^{(0)} = \lambda_{n-1} S^-\psi_n^{(0)}.$$

By comparison we obtain a recurrence relation for λ_n . It is easy to see that the constant function is an eigenfunction and moreover a lowest weight vector. Thus as the result of the analysis we find all eigenfunctions and eigenvalues.

6.2 The spectrum of a generalized YB operator

We turn to the 6-point correlator Φ_6 (3.12) obeying (2.2) with the monodromy (3.11).

We relabel the points as $1, 2, 3, 4, 5, 6 \rightarrow 1', 2', 3', 1, 2, 3$ and consider the integral operator acting as (5.11). We have shown above that this operator obeys the generalized YB relation (5.12),

$$\hat{\mathbb{R}}_{321}(\varepsilon_4, \varepsilon_5) L_3^+(u) L_2^+(\varepsilon_5 + u) L_1^+(\varepsilon_4 + \varepsilon_5 + u) = L_1^+(\varepsilon_4 + \varepsilon_5 + u) L_2^+(\varepsilon_5 + u) L_3^+(u) \hat{\mathbb{R}}_{321}(\varepsilon_4, \varepsilon_5).$$

We transform to the ratio coordinate form, where the L operators are substituted as (2.14), abbreviate the relations as

$$\hat{\mathbb{R}}_{321} \mathcal{L}(u) = \mathcal{R}(u) \hat{\mathbb{R}}_{321},$$

and expand

$$\mathcal{L}(u) = Iu^3 + u^2 \mathcal{L}^{[2]} + u \mathcal{L}^{[1]} + \mathcal{L}^{[0]}, \quad \mathcal{R}(u) = Iu^3 + u^2 \mathcal{R}^{[2]} + u \mathcal{R}^{[1]} + \mathcal{R}^{[0]}.$$

The expansion in powers of u results at u^2 in the global symmetry condition

$$[(S_1^a + S_2^a + S_3^a), \hat{\mathbb{R}}_{321}] = 0.$$

It implies that the eigenvectors of $\hat{\mathbb{R}}_{321}$ build representations with the lowest weight vectors obeying

$$(S_1^- + S_2^- + S_3^-) \psi_n^{(0)} = 0, \quad (S_1^0 + S_2^0 + S_3^0) \psi_n^{(0)} = [n - \ell_1 - \ell_2 - \ell_3] \psi_n^{(0)}, \quad (6.3)$$

and appearing explicitly as linear combinations at fixed $n = n_1 + n_2$

$$\psi_n^{(0),i} = \sum_{n_1=0}^n r(n)_{n_1}^i \psi_{n_1, n-n_1}^{(0)}, \quad (6.4)$$

of the basis lowest weight functions

$$\psi_{n_1, n_2}^{(0)} = (x_1 - x_2)^{n_1} \left(\frac{x_1 + x_2}{2} - x_3 \right)^{n_2}. \quad (6.5)$$

The problem reduces to find the particular linear combination obeying the eigenvalue relation

$$\hat{\mathbb{R}}_{321} \psi_n^{(0),i} = \lambda_n^{(i)} \psi_n^{(0),i}. \quad (6.6)$$

The degeneracy space of lowest weight vectors at given n is $n + 1$ dimensional.

Because of the global symmetry it is enough to consider one matrix element of this relation; it is convenient to choose the matrix element 12. We shall calculate the expansion coefficients in

$$\mathcal{R}_{12}^{[s]} \psi_{n_1, n-n_1}^{(0)} = \sum_k A_{+,k}^{[s]}(n-1, n_1) \psi_{n_1+k, n-n_1-k-1}^{(0)} \quad (6.7)$$

and similar in $\mathcal{L}_{12}^{[s]}$ with $A_{-,k}^{[s]}(n-1, n_1)$, at u^s , $s = 1, 0$.

We substitute (6.4) into (6.6) and use the result of the action of \mathcal{R}_{12} and \mathcal{L}_{12} .

$$\hat{\mathbb{R}}_{321} \sum_k A_{-,k} r(n)_{n_1}^i \psi_{n_1+k, n-1-n_1-k}^{(0)} = \sum_k A_{+,k} r(n)_{n_1}^i \lambda_n^{(i)} \psi_{n_1+k, n-1-n_1-k}^{(0)}.$$

The action of $\hat{\mathbb{R}}_{321}$ has been calculated on r.h.s. by the eigenvalue relation (6.6). For doing this also on l.h.s. we express on both sides $\psi_{n_1+k, n-1-n_1-k}^{(0)}$ in the eigenfunction basis by the inversion of the linear transformation (6.4).

$$\psi_{n_1, n-n_1}^{(0)} = \sum_i (r^{-1}(n))_i^{n_1} \psi_n^{(0), i}.$$

The coefficients at the eigenfunctions $\psi_{n-1}^{(0), j}$ obey the matrix equation (with the matrix elements labelled by $i = 0, \dots, n$ and $j = 0, 1, \dots, n-1$)

$$\sum_{k, n_1} A_{-,k} (r(n))_{n_1}^i (r^{-1}(n-1))_j^{n_1+k} \lambda_{n-1}^{(j)} = \sum_{k, n_1} A_{+,k} (r(n))_{n_1}^i (r^{-1}(n-1))_j^{n_1+k} \lambda_n^{(i)}.$$

We multiply by $(r^{-1}(n))_i^p (r(n-1))_q^j$ and sum over $i = 0, \dots, n$ and $j = 0, \dots, n-1$.

$$\begin{aligned} \sum_{k, j, n_1} \delta_{n_1}^p \left((r^{-1}(n-1))_j^{n_1+k} \lambda_{n-1}^{(j)} (r(n-1))_q^j \right) A_{-,k} (n-1, n_1) \\ = \sum_{k, i, n_1} A_{+,k} (n-1, n_1) \left((r^{-1}(n))_i^p \lambda_n^{(i)} (r(n))_{n_1}^i \right) \delta_q^{n_1+k}. \end{aligned} \quad (6.8)$$

We put the eigenvalues $\lambda_n^{(i)}$ at fixed n into the diagonal of a $n+1$ dimensional matrix and introduce the matrix obtained by similarity transformation with the matrix $r(n)$,

$$\Lambda(n) = r^{-1}(n) \cdot \text{diag}(\lambda_n) \cdot r(n). \quad (6.9)$$

Further, we introduce the rectangular $(n+1) \times n$ dimensional matrices \hat{A}_{\pm} with the elements

$$(\hat{A}_{\pm})_q^p = \sum_k \delta_{q-k}^p A_{\pm, k} (n-1, q-k). \quad (6.10)$$

Then the relation reads

$$(\hat{A}_{-})_{q_1}^p (\Lambda(n-1))_{q_1}^{q_1} = (\Lambda(n))_{p_1}^p (\hat{A}_{+})_{q_1}^{p_1} \quad (6.11)$$

$p, p_1 = 0, \dots, n$ and $q, q_1 = 0, \dots, n-1$.

We have $n(n+1)$ conditions at level n . These conditions with both substitutions $\hat{A}_{\pm} \rightarrow \hat{A}_{\pm}^{[s]}$, $s = 0, 1$, (6.7) are needed to find the $(n+1) \times (n+1)$ matrix elements of $\Lambda(n)$.

The matrices $\hat{A}_{\pm}^{[s]}$ encode the details of the Yangian symmetry. The matrix $r(n)$ transforms the basis $\psi_{n_1, n-n_1}^{(0)}$ to the eigenfunction basis $\psi_n^{(0), i}$.

The iterative condition (6.11) allows to calculate the matrices $\Lambda(n)$ with $\Lambda(0) = \lambda_0$ as input. The diagonalization of this matrix then results in the eigenfunctions and eigenvalues in the subspace of the irreducible representations in the threefold tensor product with weight $2\ell_n = 2\ell_1 + 2\ell_2 + 2\ell_3 - n$, at the level $n = n_1 + n_2$.

By calculating the action of the operators \mathcal{R}, \mathcal{L} appearing in the relation (5.12) at u^1 and u^0 on the basis (6.5) we shall obtain in Appendix D the explicit forms of the matrices $\hat{A}_\pm^{[1]}$ and $\hat{A}_\pm^{[0]}$.

As the result we find that they have the following non-vanishing elements, at u^1

$$\begin{aligned} (\hat{A}_\pm^{[1]})_q^q &= -(n-q) \left\{ \left(\frac{1}{2}\varepsilon_4 + \varepsilon_5 \right) \pm \left(n-1 - \frac{1}{2}\ell_1 - \frac{3}{2}\ell_2 - \ell_3 \right) \right\}, \\ (\hat{A}_\pm^{[1]})_q^{q+1} &= -(q+1) \{ \varepsilon_4 \pm (q - \ell_1 - \ell_2) \}, \\ (\hat{A}_\pm^{[1]})_q^{q-1} &= \pm \frac{1}{4}(n-q+1)(n-q), \end{aligned} \tag{6.12}$$

and at u^0

$$\begin{aligned} (\hat{A}_\pm^{[0]})_q^q &= (n-q) \left\{ \left[-\ell_1\ell_2 - \frac{1}{2}q(q-1-\ell_1-3\ell_2) + \frac{1}{2}(\ell_1-\ell_2)(n-q-1-\ell_3) \right] \right. \\ &\quad \left. \mp \left[\varepsilon_5(n-1-\ell_1-\ell_2-\ell_3) + \frac{1}{2}\varepsilon_4(n-1-\ell_3-2\ell_2) \right] - \varepsilon_5(\varepsilon_4 + \varepsilon_5) \right\}, \\ (\hat{A}_\pm^{[0]})_q^{q+1} &= (q+1) \left\{ (q-\ell_1-\ell_2)(n-q-1-\ell_3) \pm \varepsilon_4(n-q-1-\ell_3) \right\}, \\ (\hat{A}_\pm^{[0]})_q^{q-1} &= \frac{1}{4}(n-q+1)(n-q) \left\{ q-1-n-\ell_1+3\ell_2+\ell_3 \pm \varepsilon_4 \right\}, \\ (\hat{A}_\pm^{[0]})_q^{q-2} &= \frac{1}{8}(n-q+2)(n-q+1)(n-q). \end{aligned} \tag{6.13}$$

A convenient algorithm is based on the observation that the matrices \hat{A}_\pm are near diagonal. A quadratic diagonal matrix $Z(n)$ can be found,

$$Z(n)_{qq} = \ell_3 - n + q + 1, \quad q = 0, 1, \dots, n-1,$$

such that $\hat{A}_+^{[1]}Z(n) + \hat{A}_+^{[0]} = \hat{A}'_+$ is upper triangular and has zero elements on the last row. The two conditions (6.11) with $\hat{A}_\pm^{[1]}$ and $\hat{A}_\pm^{[0]}$ can be combined in such a way that \hat{A}'_+ appears on r.h.s. This linear combination of the conditions allows to calculate easily the matrix elements $\Lambda(n)_{p_1}^p$ with $p = 0, 1, \dots, n$ and $p_1 = 0, 1, \dots, n-1$. After this the matrix $\Lambda(n)$ with unknown elements on the last column only is substituted in one of the initial conditions, e.g. in the one involving $\hat{A}_\pm^{[1]}$ (6.12), to calculate these remaining elements.

7 Discussion

The approach of Yangian symmetric correlators provides a simple way to treat representations of the Yangian algebra in particular in the case of Jordan-Schwinger type realizations of sl_n . The considered case of sl_2 has the advantage that several explicit forms of the relevant Yang-Baxter operators are known. This provides flexibility in the calculations and allows formulations beyond those parallel to the methods developed for scattering amplitudes. It is convenient to use the forms of the L and R operators both in homogeneous and in ratio coordinates. If the ratio coordinate x describes position by transformation to the helicity form one obtains the corresponding momentum dependence ($k = \lambda\bar{\lambda}$).

The monodromy matrix includes in the expansion in the spectral parameter u ($u_i = u + \delta_i$) the generators of the considered Yangian algebra representation. The latter is characterized by the conformal weights $2\ell_i$ and the shifts δ_i , which are the differences of the spectral parameters in $L_i(u_i)$, $u_{i+1} - u_i$. The YSC defined by the monodromy depends on these representation parameters. The described R operator construction results in the weights calculated as spectral parameter differences as well, moreover the sum of all weights is fixed by the elementary correlators from which the construction starts. This means that we deal with particular representations where the representation parameters are not all independent. We have encountered examples with more than one relation constraining the weights. In such cases not all spectral parameter differences or shifts are fixed by the weights, some differences appear as extra parameters. These extra parameters together with the independent weights characterize the representation and the related YSC. In the considered example of a 6-point YSC we have 2 extra parameters. This example can be generalized to $2M$ -point YSC with $M - 1$ extra parameters.

The Yangian symmetry condition (2.2) in terms of the monodromy matrix allows generalizations, where the L matrices in the latter are replaced by the ones with trigonometric or elliptic deformations or by other forms related to the standard one by Drinfeld twists. It would be interesting to use such opportunities in applications beyond the well studied integrable models.

The construction of YSC starting from elementary correlators relies on R operators obeying Yang-Baxter relations with the product of two L operators. These relations imply simple rules for the interchange of the R operators with the product of L operators in the monodromy.

Actually the L operators, which have been written in several forms, are the source of the rich symmetry structures. The simplicity of the L operator in the case of sl_2 allows to write explicitly several kinds and representations of R operators.

We have considered symmetric operators with YSC as kernels. The Yangian symmetry of the kernel implies the symmetry of the operator. This follows from the L operators by their relations of inversion and transposition.

Particular 4-point YSC result in the different kinds of YB operators. The considered 6-point YSC results in a generalized YB operator and the mentioned generalization of the latter to $2M$ points lead to generalized YB operators where the counterpart of (5.12) involves the products of M factors of L matrices on both sides. The spectral parameter arguments of the generalized YB operators are related to the extra parameters in their kernels.

As a tool of treating Yangian symmetry the YSC approach is expected to be useful in physical applications. In this respect it is important that we have seen explicitly how the symmetry of the correlators implies the symmetry of the operators constructed out of them and how the high symmetry allows to solve exactly the spectral problem. We have formulated this in the non-trivial case of a 3-point generalized YB operators constructed from a particular YSC. The outlined procedure generalizes to more points with increasing complexity.

As an example of applications we have shown the direct relation of particular 4-point YSC to the kernels of the scale evolution of (generalized) parton distributions in leading order QCD. They are the ones appearing as kernels of the YB operator $\mathbb{R}_{12}(v)$ intertwining the tensor product of the representations with the weights $2\ell_1, 2\ell_2$. We have identified the values of the weights and of the parameter v appropriate for different cases of parton distributions, of gluons or quarks and of different polarization configurations. In QCD

the parton kernels are sums of contributions of 4-point YSC with different values of the additional parameter.

These 4-point kernels describe simultaneously the scale dependence of a class of composite operators, actually the ones composed of two quasi-partonic field operators of gluons or quarks. The considered 6-point YSC is to be applied to the scale dependence of composite operators out of 3 quark or gluon field operators.

In this way we have a reformulation in terms of YSC of the known relation of the scale dependence of composite operators to the dynamics of quantum spin chains.

The successful integrable spin chain treatment of all composite operators (with an arbitrary number of fields) of $\mathcal{N} = 4$ super Yang-Mills is related to the fact that here in the renormalization of analogous composite operators only the contribution of one value of the additional parameter, $v \rightarrow 0$, appears. Whereas in QCD in general more than one Yangian representation contributes, here one encounters only one contribution.

It would be interesting to consider the YSC based on the $sl(4|4)$ superalgebra for a unified treatment of the quantum integrability properties of $\mathcal{N} = 4$ super Yang-Mills, i.e. to have on one hand the known way to the scattering amplitudes and on the other hand an alternative way to the treatment of the operator product renormalization.

Acknowledgments

The authors are grateful to A.P. Isaev and Č. Burdík for discussions and support.

Our work has been supported by JINR Dubna via a Heisenberg-Landau grant. The work of J.F. has been supported by the Grant Agency of the Czech Technical University in Prague, grant No. *SGS15/215/OHK4/3T/14* and by the Grant of the Plenipotentiary of the Czech republic at JINR, Dubna.

A YB operator R^{++} as contour integral

We want to obey the RLL relation with the ansatz

$$R_{12}^{++}(u) = \int dc \phi(c) e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)}.$$

We calculate the action of the shift operator on the product of L matrices:

$$e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)} \mathbf{p}_1 = (\mathbf{p}_1 + c\mathbf{p}_2) e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)}, \quad e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)} \mathbf{x}_2 = (\mathbf{x}_2 - c\mathbf{x}_1) e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)},$$

$$\begin{aligned} e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)} L_1^+(u) L_2^+(v) &= (L_1^+(u) + c\mathbf{p}_2 \otimes \mathbf{x}_1) (L_2^+(v) - c\mathbf{p}_2 \otimes \mathbf{x}_1) e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)} \\ &= \left(L_1^+(v) L_2^+(u) + [L_2^+(0) - L_1^+(0)] [u - v + c(\mathbf{x}_1 \cdot \mathbf{p}_2)] - c[u - v - 1 + c(\mathbf{x}_1 \cdot \mathbf{p}_2)] \right) e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)}. \end{aligned}$$

The condition that the second term vanishes implies a differential equation on $\phi(c)$, because

$$\int dc \phi(c) \cdot c \cdot (\mathbf{x}_1 \cdot \mathbf{p}_2) e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)} = - \int dc \phi(c) \cdot c \cdot \partial_c e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)} = \int dc \partial_c (c\phi(c)) e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)}.$$

In the last step we have assumed that the integration by parts is done without boundary terms. Thus the condition on ϕ is

$$\partial_c (c\phi(c)) + (u - v)\phi(c) = 0.$$

It is solved by

$$\phi(c) = \frac{1}{c^{1+u-v}}.$$

The condition of vanishing of the third term can be written as

$$0 = \partial_c(c^2\phi(c)) + c(u-v-1)\phi(c) = c[\partial_c(c\phi(c)) + (u-v)\phi(c)].$$

We see that it does not imply a further condition on $\phi(c)$. Thus we have proved that the form used in calculation obeys the Yang-Baxter relation provided the simple rule of integration by parts. If the latter rule is different then the indicated procedure leads to the appropriate modification.

B Calculation of the 6-point correlator

We provide details of calculations leading to the correlator (3.12). The δ -functions can be used to fix six out of the nine integration variables as the solution of ($i = 1, 2, 3$)

$$x_{i,1} - c_{i4}x_{4,1} - c_{i5}x_{5,1} - c_{i6}x_{6,1} = 0, \quad x_{i,2} - c_{i4}x_{4,22} - c_{i5}x_{5,2} - c_{i6}x_{6,2} = 0.$$

We let c_{i6} free and find c_{i4}, c_{i5} and the Jacobi factor $\langle 45 \rangle^{-3}$,

$$c_{i4}^{(0)} = \frac{\langle i5 \rangle - c_{i6}\langle 65 \rangle}{\langle 45 \rangle}, \quad c_{i5}^{(0)} = \frac{\langle i4 \rangle - c_{i6}\langle 64 \rangle}{\langle 54 \rangle},$$

$$\prod_{i=1}^3 \delta^{(2)}(\mathbf{x}_i - \sum_{j=4}^6 c_{ij}\mathbf{x}_j) = \langle 45 \rangle^{-3} \prod_{i=1}^3 \delta(c_{i4} - c_{i4}^{(0)}) \delta(c_{i5} - c_{i5}^{(0)}).$$

The minors result in

$$(234) \rightarrow c_{14}^{(0)} = \frac{\langle 15 \rangle - c_{16}\langle 65 \rangle}{\langle 45 \rangle},$$

$$(345) \rightarrow \begin{vmatrix} c_{14}^{(0)} & c_{15}^{(0)} \\ c_{24}^{(0)} & c_{25}^{(0)} \end{vmatrix} = \langle 45 \rangle^{-2} \begin{vmatrix} \langle 15 \rangle - c_{16}\langle 65 \rangle & \langle 14 \rangle - c_{16}\langle 64 \rangle \\ \langle 25 \rangle - c_{26}\langle 65 \rangle & \langle 24 \rangle - c_{26}\langle 64 \rangle \end{vmatrix}.$$

We use relations of the type

$$\begin{vmatrix} \langle 15 \rangle & \langle 14 \rangle \\ \langle 25 \rangle & \langle 24 \rangle \end{vmatrix} = \langle 12 \rangle \langle 54 \rangle$$

and obtain

$$(345) \rightarrow \langle 45 \rangle^{-1} (\langle 12 \rangle - c_{16}\langle 62 \rangle + c_{26}\langle 61 \rangle),$$

$$(456) \rightarrow \langle 45 \rangle^{-2} \begin{vmatrix} c_{14}^{(0)} & c_{15}^{(0)} & c_{16} \\ c_{24}^{(0)} & c_{25}^{(0)} & c_{26} \\ c_{34}^{(0)} & c_{35}^{(0)} & c_{36} \end{vmatrix} = \langle 45 \rangle^{-1} (\langle 12 \rangle c_{36} + \langle 23 \rangle c_{16} + \langle 31 \rangle c_{26}),$$

$$(561) \rightarrow \langle 45 \rangle^{-1} \begin{vmatrix} c_{25}^{(0)} & c_{26} \\ c_{35}^{(0)} & c_{36} \end{vmatrix} = \langle 45 \rangle^{-1} (\langle 24 \rangle c_{36} - \langle 34 \rangle c_{26}),$$

$$(612) \rightarrow c_{36}.$$

C q-deformed YB operator \mathcal{S}_{q12}

We use the abbreviation

$$\Phi(x, \lambda) = \frac{(xq^{1-\lambda}; q^2)}{(xq^{1+\lambda}; q^2)}$$

defined using the infinite product $(x; q) := \prod_{j=1}^{\infty} (1 - xq^{j-1})$. Because $\mathcal{S}_{q12}(w)$ involves the coordinates only, it commutes with the matrix factors \hat{V}_i in the factorization (4.3). Commuting it through the matrix factors \hat{D}_i creates extra matrix factors. Commuting $\mathcal{S}_{q12}(v_1^{(2)} - v_2^{(1)})$ through $L_1(v_1^{(1)}, v_1^{(2)})L_2(v_2^{(1)}, v_2^{(2)})$ we see that it commutes with the first factor of L_1 and the third factor of L_2 . We get from the second and third factor of L_1 and the first and second factor of L_2

$$\begin{aligned} & \begin{pmatrix} q^{-v_1^{(2)}+v_2^{(1)}} \frac{\Phi(qx_2/x_1, v_1^{(2)}-v_2^{(1)})}{\Phi(x_2/x_1, v_1^{(2)}-v_2^{(1)})} & 0 \\ 0 & q^{v_1^{(2)}-v_2^{(1)}} \frac{\Phi(q^{-1}x_2/x_1, v_1^{(2)}-v_2^{(1)})}{\Phi(x_2/x_1, v_1^{(2)}-v_2^{(1)})} \end{pmatrix} \begin{pmatrix} q^{v_1^{(2)}} & x_1^{-1} \\ -q^{-v_1^{(2)}} & -x_1^{-1} \end{pmatrix} \\ & \quad \times \begin{pmatrix} 1 & 1 \\ -x_2q^{-v_2^{(1)}} & -x_2q^{v_2^{(1)}} \end{pmatrix} \begin{pmatrix} \frac{\Phi(x_2/x_1, v_1^{(2)}-v_2^{(1)})}{\Phi(qx_2/x_1, v_1^{(2)}-v_2^{(1)})} & 0 \\ 0 & \frac{\Phi(x_2/x_1, v_1^{(2)}-v_2^{(1)})}{\Phi(q^{-1}x_2/x_1, v_1^{(2)}-v_2^{(1)})} \end{pmatrix} \\ & = \begin{pmatrix} q^{v_2^{(1)}} - \frac{x_2}{x_1}q^{-v_1^{(2)}} & q^{v_2^{(1)}} - \frac{x_2}{x_1}q^{v_1^{(2)}} \\ \frac{x_2}{x_1}q^{-v_1^{(2)}} - q^{-v_2^{(1)}} & \frac{x_2}{x_1}q^{v_1^{(2)}} - q^{-v_2^{(1)}} \end{pmatrix} = \begin{pmatrix} q^{v_2^{(1)}} & x_1^{-1} \\ -q^{-v_2^{(1)}} & -x_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -x_2q^{-v_1^{(2)}} & -x_2q^{v_1^{(2)}} \end{pmatrix} \end{aligned}$$

which are the third factor of L_1 and the first factor of L_2 on the right hand side of (4.8).

The proof can be repeated also in terms of the homogeneous coordinates $x_{i,1}, x_{i,2}$. The L^+ operator (4.1) can be reconstructed from the one in the ratio coordinates (4.3) by inserting $x_1 = \frac{x_{1,1}}{x_{1,2}}$ and $2\ell_1 \rightarrow x_{1,1}\partial_{1,1} + x_{1,2}\partial_{1,2}$ and in analogous way also the L^- operator.

To simplify formulas we use in the rest of this appendix the following notation: $x_1 = x_{1,1}, x_2 = x_{1,2}, y_1 = x_{2,1}, y_2 = x_{2,2}$. The derivatives appearing below are always multiplied by the respective variables and we adopt the following simplified notation: $x_1\partial_1 = x_{1,1}\partial_{1,1}, x_2\partial_2 = x_{1,2}\partial_{1,2}$, whereas $y_1\partial_1 = x_{2,1}\partial_{2,1}, y_2\partial_2 = x_{2,2}\partial_{2,2}$. The q-deformed R operator

$$R(\lambda) = (x_1y_1)^\lambda \cdot \Phi(\mathcal{X}, \lambda), \quad (\text{C.1})$$

where $\mathcal{X} = -\frac{x_2y_2}{x_1y_1}$, commutes with factors of the L operators as

$$\begin{aligned} R \begin{pmatrix} 1 & 1 \\ \frac{x_1}{x_2}q^{-u-x_1\partial_1-x_2\partial_2} & \frac{x_1}{x_2}q^{u+x_1\partial_1+x_2\partial_2} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ \frac{x_1}{x_2}q^{-u+\lambda-x_1\partial_1-x_2\partial_2} & \frac{x_1}{x_2}q^{u-\lambda+x_1\partial_1+x_2\partial_2} \end{pmatrix} R, \\ R \begin{pmatrix} q^{x_1\partial_1+1} & 0 \\ 0 & q^{-x_1\partial_1-1} \end{pmatrix} &= \begin{pmatrix} q^{x_1\partial_1+1-\lambda} \frac{\Phi(q\mathcal{X}, \lambda)}{\Phi(\mathcal{X}, \lambda)} & 0 \\ 0 & q^{\lambda-x_1\partial_1-1} \frac{\Phi(q^{-1}\mathcal{X}, \lambda)}{\Phi(\mathcal{X}, \lambda)} \end{pmatrix} R, \\ R \begin{pmatrix} q^{y_2\partial_2+1} & 0 \\ 0 & q^{-y_2\partial_2-1} \end{pmatrix} &= \begin{pmatrix} \frac{\Phi(\mathcal{X}, \lambda)}{\Phi(q\mathcal{X}, \lambda)} q^{y_2\partial_2+1} & 0 \\ 0 & \frac{\Phi(\mathcal{X}, \lambda)}{\Phi(q^{-1}\mathcal{X}, \lambda)} q^{-y_2\partial_2-1} \end{pmatrix} R, \\ R \begin{pmatrix} q^{v-y_1\partial_1-y_2\partial_2-2} & \frac{y_1}{y_2} \\ -q^{-v+y_1\partial_1+y_2\partial_2+2} & -\frac{y_1}{y_2} \end{pmatrix} &= \begin{pmatrix} q^{v+\lambda-y_1\partial_1-y_2\partial_2-2} & \frac{y_1}{y_2} \\ -q^{-v-\lambda+y_1\partial_1+y_2\partial_2+2} & -\frac{y_1}{y_2} \end{pmatrix} R. \end{aligned}$$

After commuting $R(\lambda)$ through $L^+(u)L^-(v)$ we obtain from the second and the third factor of $L^+(u)$ and the first and the second factor of $L^-(v)$

$$\begin{aligned} & \begin{pmatrix} q^{-\lambda} \frac{\Phi(q\mathcal{X}, \lambda)}{\Phi(\mathcal{X}, \lambda)} & 0 \\ 0 & q^\lambda \frac{\Phi(q^{-1}\mathcal{X}, \lambda)}{\Phi(\mathcal{X}, \lambda)} \end{pmatrix} \begin{pmatrix} q^{u-1} & -\frac{x_2}{x_1} \\ -q^{-u+1} & \frac{x_2}{x_1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{y_2}{y_1} q^{-v+1} & -\frac{y_2}{y_1} q^{v-1} \end{pmatrix} \\ & \quad \times \begin{pmatrix} \frac{\Phi(\mathcal{X}, \lambda)}{\Phi(q\mathcal{X}, \lambda)} & 0 \\ 0 & \frac{\Phi(\mathcal{X}, \lambda)}{\Phi(q^{-1}\mathcal{X}, \lambda)} \end{pmatrix} \\ &= \begin{pmatrix} q^{u-\lambda-1} + \mathcal{X}q^{-v-\lambda+1} & (q^{-\lambda+u-1} + \mathcal{X}q^{-\lambda+v-1}) \frac{\Phi(q\mathcal{X}, \lambda)}{\Phi(q^{-1}\mathcal{X}, \lambda)} \\ -(q^{\lambda-u+1} + \mathcal{X}q^{\lambda-v+1}) \frac{\Phi(q^{-1}\mathcal{X}, \lambda)}{\Phi(q\mathcal{X}, \lambda)} & -(q^{\lambda-u+1} + \mathcal{X}q^{\lambda+v-1}) \end{pmatrix} \end{aligned}$$

and setting $\lambda = u - v$

$$= \begin{pmatrix} q^{v-1} + \mathcal{X}q^{-u+1} & q^{v-1} + \mathcal{X}q^{u-1} \\ -(q^{-v+1} \mathcal{X}q^{-u+1}) & -(q^{-v+1} + \mathcal{X}q^{u-1}) \end{pmatrix} = \begin{pmatrix} q^{v-1} & -\frac{x_2}{x_1} \\ -q^{-v+1} & \frac{x_2}{x_1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{y_2}{y_1} q^{-u+1} & -\frac{y_2}{y_1} q^{u-1} \end{pmatrix}$$

which proves (4.10).

D Calculation of \hat{A}_\pm

At u^1 in the generalized YB relation (5.12) appears

$$\begin{aligned} & \hat{\mathbb{R}} (\sigma S_3 \sigma S_2 + \varepsilon_5 \sigma S_3 + \sigma S_3 \sigma S_1 + (\varepsilon_4 + \varepsilon_5) \sigma S_3 + \sigma S_2 \sigma S_1 + \varepsilon_5 \sigma S_1 (\varepsilon_4 + \varepsilon_5) \sigma S_2) \\ &= (\sigma S_1 \sigma S_2 + (\varepsilon_4 + \varepsilon_5) \sigma S_2 + \varepsilon_5 \sigma S_1 + \sigma S_1 \sigma S_3 + (\varepsilon_4 + \varepsilon_5) \sigma S_3 + \sigma S_2 \sigma S_3 + \varepsilon_5 \sigma S_3) \hat{\mathbb{R}}. \end{aligned}$$

We have abbreviated using Pauli matrices $\sigma S = \sigma^a S^a$. We consider the matrix element 12,

$$\begin{aligned} \mathcal{R}_{12} &= \mathcal{R}_{12}^{(1)} + \mathcal{R}_{12}^{(2)}, & \mathcal{L}_{12} &= \mathcal{L}_{12}^{(1)} + \mathcal{L}_{12}^{(2)}, \\ \mathcal{R}_{12}^{(2)} &= -\mathcal{L}_{12}^{(2)}, & \mathcal{R}_{12}^{(1)} &= \mathcal{L}_{12}^{(1)}, \\ \mathcal{R}_{12}^{(1)} &= \varepsilon_4 S_2^- + (\varepsilon_4 + \varepsilon_5) S_3^-, \\ \mathcal{R}_{12}^{(2)} &= S_1^0 S_2^- - S_1^- S_2^0 + S_1^0 S_3^- - S_1^- S_3^0 + S_2^0 S_3^- - S_2^- S_3^0. \end{aligned}$$

We have suppressed the superscript $[s]$ indicating the power of expansion in u .

We calculate the action of the operators involved on the basis $\psi_{n_1, n_2}^{(0)}$.

$$\begin{aligned} \mathcal{R}_{12}^{(1)} \psi_{n_1, n_2}^{(0)} &= -\varepsilon_4 (n_1 \psi_{n_1-1, n_2}^{(0)} + \frac{1}{2} n_2 \psi_{n_1, n_2-1}^{(0)}) - \varepsilon_5 n_2 \psi_{n_1, n_2-1}^{(0)}, \\ \mathcal{R}_{12}^{(2)} \psi_{n_1, n_2}^{(0)} &= -(S_1^0 + S_2^0) (n_1 \psi_{n_1-1, n_2}^{(0)} + \frac{1}{2} n_2 \psi_{n_1, n_2-1}^{(0)}) \\ &\quad - (S_1^- + S_2^-) (-\ell_3 \psi_{n_1, n_2}^{(0)} - n_2 x_3 - \psi_{n_1, n_2-1}^{(0)}) - S_2^0 n_2 \psi_{n_1, n_2-1}^{(0)} \\ &= \psi_{n_1, n_2-1}^{(0)} \left(-\frac{1}{2} n_2 (n_1 + n_2 - 1 - \ell_1 - \ell_2) + n_2 (\ell_1 + \ell_2) \right) \\ &\quad + \psi_{n_1-1, n_2}^{(0)} (-n_1 (n_1 + n_2 - 1 - \ell_1 - \ell_2)) + \psi_{n_1-1, n_2-1}^{(0)} (-n_1 n_2 x_3 + n_1 n_2 x_2) \\ &\quad + \psi_{n_1, n_2-2}^{(0)} \left(-\frac{1}{2} n_2 (n_2 - 1) x_3 + n_2 (n_2 - 1) x_3 - n_2 (n_2 - 1) x_2 \right). \quad (\text{D.2}) \end{aligned}$$

We use

$$(x_3 - x_2)\psi_{n_1, n_2}^{(0)} = \frac{1}{2}\psi_{n_1+1, n_2}^{(0)} - \psi_{n_1, n_2+1}^{(0)}$$

and obtain

$$\begin{aligned} \mathcal{R}_{12}^{(2)}\psi_{n_1, n_2}^{(0)} &= -n_2\left(n_1 + n_2 - 1 - \frac{1}{2}\ell_1 - \frac{3}{2}\ell_2 - \ell_3\right)\psi_{n_1, n_2-1}^{(0)} \\ &\quad - n_1(n_1 - 1 - \ell_1 - \ell_2)\psi_{n_1-1, n_2}^{(0)} + \frac{1}{4}n_2(n_2 - 1)\psi_{n_1+1, n_2-2}^{(0)}. \end{aligned} \quad (\text{D.3})$$

The result of the operator action at u^1 can be formulated in terms of the matrices $\hat{A}_{\pm}^{[1]}$ (6.10) as written above (6.12).

We consider also the u^0 contribution to (5.12). With the abbreviation $\mathcal{RL} = \mathcal{RR}$ we have

$$\begin{aligned} \mathcal{L} &= (\sigma S_3)(\sigma S_2)(\sigma S_1) + \varepsilon_5(\sigma S_3)(\sigma S_1) + (\varepsilon_4 + \varepsilon_5)(\sigma S_3)(\sigma S_2) + \varepsilon_5(\varepsilon_4 + \varepsilon_5)(\sigma S_3), \\ \mathcal{R} &= (\sigma S_1)(\sigma S_2)(\sigma S_3) + \varepsilon_5(\sigma S_1)(\sigma S_3) + (\varepsilon_4 + \varepsilon_5)(\sigma S_2)(\sigma S_3) + \varepsilon_5(\varepsilon_4 + \varepsilon_5)(\sigma S_3). \end{aligned}$$

We shall use again the matrix element 12 of the relation.

$$\begin{aligned} \mathcal{R} &= \mathcal{R}^{(3)} + \mathcal{R}^{(2)} + \mathcal{R}^{(1)}, \\ \mathcal{R}_{12}^{(3)} &= S_1^0 S_2^0 S_3^- + S_1^- S_2^+ S_3^- - S_1^0 S_2^- S_3^0 + S_1^- S_2^0 S_3^0, \\ \mathcal{R}^{(2)} &= \varepsilon_5(S_1^0 S_3^- - S_1^- S_3^0) + (\varepsilon_4 + \varepsilon_5)(S_2^0 S_3^- - S_2^- S_3^0), \\ \mathcal{R}^{(1)} &= \varepsilon_5(\varepsilon_4 + \varepsilon_5)S_3^-. \end{aligned}$$

We suppress the superscript $[s]$ indicating the expansion power in u . Similar to the case u^1 we have

$$\mathcal{L}^{(3)} = \mathcal{R}^{(3)}, \quad \mathcal{L}^{(2)} = -\mathcal{R}^{(2)}, \quad \mathcal{L}^{(1)} = \mathcal{R}^{(1)}.$$

We calculate the action of these operators on the basis functions of lowest weight (6.5). We start with contributions to $\mathcal{R}_{12}^{(3)}$.

$$\begin{aligned} (S_1^0 S_2^0 + S_1^- S_2^+)\psi_{n_1, n_2}^{(0)} &= \{(x_1 x_2 - x_2^2)\partial_1 \partial_2 - \ell_1 x_2 \partial_2 - \ell_2 x_1 \partial_1 + 2\ell_2 x_2 \partial_1 + \ell_1 \ell_2\}\psi_{n_1, n_2}^{(0)} \\ &= x_2(x_1 - x_2)\left\{-n_1(n_1 - 1)\psi_{n_1-2, n_2}^{(0)} + \frac{1}{4}n_2(n_2 - 1)\psi_{n_1, n_2-2}^{(0)}\right\} + \ell_1 \ell_2 \psi_{n_1, n_2}^{(0)} \\ &\quad + \ell_2(2x_2 - x_1)\left\{n_1\psi_{n_1-1, n_2}^{(0)} + \frac{1}{2}n_2\psi_{n_1, n_2-1}^{(0)}\right\} - \ell_1 x_2\left\{-n_1\psi_{n_1-1, n_2}^{(0)} + \frac{1}{2}n_2\psi_{n_1, n_2-1}^{(0)}\right\}. \end{aligned}$$

We substitute

$$x_2 = \left(\frac{x_1 + x_2}{2} - x_3\right) - \frac{1}{2}(x_1 - x_2) + x_3$$

and separate a contribution proportional to x_3 .

$$\begin{aligned} (S_1^0 S_2^0 + S_1^- S_2^+)\psi_{n_1, n_2}^{(0)} &= x_3\left\{(-n_1 + 1 + \ell_1 + \ell_2)n_1\psi_{n_1-1, n_2}^{(0)} + \frac{1}{4}n_2(n_2 - 1)\psi_{n_1+1, n_2-2}^{(0)} - (\ell_1 - \ell_2)\frac{1}{2}n_2\psi_{n_1, n_2-1}^{(0)}\right\} \\ &\quad + (-n_1 + 1 + \ell_1 + \ell_2)n_1\psi_{n_1-1, n_2+1}^{(0)} + \left\{\frac{1}{2}(n_2 - 1) - \ell_2 + \frac{1}{2}(\ell_1 - \ell_2)\right\}\frac{1}{2}n_2\psi_{n_1+1, n_2-1}^{(0)} \\ &\quad - \frac{1}{8}n_2(n_2 - 1)\psi_{n_1+2, n_2-2}^{(0)} + \left\{\ell_1 \ell_2 + \frac{1}{2}n_1(n_1 - 1) - n_1\left(\frac{1}{2}\ell_1 + \frac{3}{2}\ell_2\right) - \frac{1}{2}n_2(\ell_1 - \ell_2)\right\}\psi_{n_1, n_2}^{(0)}, \end{aligned}$$

$$\begin{aligned}
& S_3^-(S_1^0 S_2^0 + S_1^- S_2^+) \psi_{n_1, n_2}^{(0)} \\
&= -x_3 \left\{ (-n_1 + 1 + \ell_1 + \ell_2) n_1 n_2 \psi_{n_1-1, n_2-1}^{(0)} + \frac{1}{4} n_2 (n_2 - 1) (n_2 - 2) \psi_{n_1+1, n_2-3}^{(0)} \right. \\
&\quad \left. - (\ell_1 - \ell_2) \frac{1}{2} n_2 (n_2 - 1) \psi_{n_1, n_2-2}^{(0)} \right\} + (n_1 - 1 - \ell_1 - \ell_2) n_1 n_2 \psi_{n_1-1, n_2}^{(0)} \\
&\quad + \frac{1}{8} n_2 (n_2 - 1) (n_2 - 2) \psi_{n_1+2, n_2-3}^{(0)} - (n_2 - 2 + \ell_1 - 3\ell_2) \frac{1}{4} n_2 (n_2 - 1) \psi_{n_1+1, n_2-2}^{(0)} \\
&\quad - \left\{ 2\ell_1 \ell_2 + n_1 (n_1 - 1) - n_1 (\ell_1 + 3\ell_2) - (n_2 - 1) (\ell_1 - \ell_2) \right\} \frac{1}{2} n_2 \psi_{n_1, n_2-1}^{(0)},
\end{aligned}$$

$$\begin{aligned}
& (S_1^0 S_2^- - S_1^- S_2^0) \psi_{n_1, n_2}^{(0)} \\
&= (-n_1 + 1 + \ell_1 + \ell_2) n_1 \psi_{n_1-1, n_2}^{(0)} + \frac{1}{4} n_2 (n_2 - 1) \psi_{n_1+1, n_2-2}^{(0)} - (\ell_1 - \ell_2) \frac{1}{2} n_2 \psi_{n_1, n_2-1}^{(0)},
\end{aligned}$$

$$\begin{aligned}
& S_3^0 (S_1^0 S_2^- - S_1^- S_2^0) \psi_{n_1, n_2}^{(0)} \\
&= -x_3 \left\{ (-n_1 + 1 + \ell_1 + \ell_2) n_1 n_2 \psi_{n_1-1, n_2-1}^{(0)} \right. \\
&\quad \left. + \frac{1}{4} n_2 (n_2 - 1) (n_2 - 2) \psi_{n_1+1, n_2-3}^{(0)} - (\ell_1 - \ell_2) \frac{1}{2} n_2 (n_2 - 1) \psi_{n_1, n_2-2}^{(0)} \right\} \\
&\quad - \ell_3 \left\{ (-n_1 + 1 + \ell_1 + \ell_2) n_1 \psi_{n_1-1, n_2}^{(0)} + \frac{1}{4} n_2 (n_2 - 1) \psi_{n_1+1, n_2-2}^{(0)} - (\ell_1 - \ell_2) \frac{1}{2} n_2 \psi_{n_1, n_2-1}^{(0)} \right\}.
\end{aligned}$$

In the action of $\mathcal{R}_{12}^{(3)} = (S_1^0 S_2^0 + S_1^- S_2^+) S_3^- - (S_1^0 S_2^- + S_1^- S_2^0) S_3^0$ the contributions proportional to x_3 cancel. It is important to find the result containing lowest weight contributions only.

$$\begin{aligned}
& \mathcal{R}_{12}^{(3)} \psi_{n_1, n_2}^{(0)} \\
&= n_1 (n_1 - 1 - \ell_1 - \ell_2) (n_2 - \ell_3) \psi_{n_1-1, n_2}^{(0)} + \frac{1}{8} n_2 (n_2 - 1) (n_2 - 2) \psi_{n_1+2, n_2-3}^{(0)} \\
&\quad + n_2 \left\{ -\frac{1}{2} (\ell_1 - \ell_2) (\ell_3 + 1) - \ell_1 \ell_2 - \frac{1}{2} n_1 (n_1 - 1 - \ell_1 - 3\ell_2) + \frac{1}{2} n_2 (\ell_1 - \ell_2) \right\} \psi_{n_1, n_2-1}^{(0)} \\
&\quad - \frac{1}{4} n_2 (n_2 - 1) (n_2 + \ell_1 - 3\ell_2 - \ell_3) \psi_{n_1+1, n_2-2}^{(0)}. \quad (\text{D.4})
\end{aligned}$$

The action of $\mathcal{R}^{(2)}$ is calculated easier.

$$\begin{aligned}
& \mathcal{R}_{12}^{(2)} \psi_{n_1, n_2}^{(0)} = \frac{1}{4} n_2 (n_2 - 1) \varepsilon_4 \psi_{n_1+1, n_2-2}^{(0)} + \varepsilon_4 n_1 (n_2 - \ell_3) \psi_{n_1-1, n_2}^{(0)} \\
&\quad - n_2 \left\{ \varepsilon_5 (n_1 + n_2 - 1 - \ell_1 - \ell_2 - \ell_3) + \frac{1}{2} \varepsilon_4 (n_1 + n_2 - 1 - \ell_3 - 2\ell_2) \right\} \psi_{n_1, n_2-1}^{(0)}. \quad (\text{D.5})
\end{aligned}$$

We add also

$$\mathcal{R}_{12}^{(1)} \psi_{n_1, n_2}^{(0)} = -\varepsilon_5 (\varepsilon_4 + \varepsilon_5) n_2 \psi_{n_1, n_2-1}^{(0)}. \quad (\text{D.6})$$

According to our conventions the result can be written in terms of the matrices $\hat{A}_{\pm}^{[0]}$ given above (6.13).

References.

- [1] V. G. Drinfeld, “*Hopf algebras and the quantum Yang-Baxter equation*”, Sov. Math. Dokl. **32** (1985) 254 [Dokl. Akad. Nauk Ser. Fiz. **283** (1985) 1060]; “*A New realization of Yangians and quantized affine algebras*”, Sov. Math. Dokl. **36** (1988) 212.
- [2] L. N. Lipatov, “High-energy asymptotics of multicolor QCD and exactly solvable lattice models”, JETP Lett. 59(1994) 596, Padua preprint DFPD/93/TH/70, hep-th/9311037.
- [3] V.M. Braun, S.E. Derkachov, A.N. Manashov, Phys. Rev. Lett 81 (1998) 2020
- [4] L.N. Lipatov, Sov.J.Nucl.Phys. 23(1976)338
V.S. Fadin, E.A. Kuraev and L.N. Lipatov, Phys. Lett. 60B(1975)50; Sov.Phys. JETP 44(1976)443;
ibid 45(1977)199
Y.Y. Balitski and L.N. Lipatov, Sov.J.Nucl.Phys. 28(1978)882.
- [5] V.G. Gribov and L.N. Lipatov, Sov. J. Nucl. Phys. 15(1972)438
L.N. Lipatov, Yad. Fiz. 20(1974)532
G. Altarelli and G. Parisi, Nucl. Phys. B126(1977)298
Yu.L. Dokshitzer, ZhETF 71(1977)1216
- [6] V.L. Chernyak and A.R. Zhitnitsky, JETP Lett 25 (1977) 510;
A.V. Efremov, A.V. Radyushkin, Theor. Math. Phys. 42 (1980) 97; Phys. Lett. B94 (1980) 245.
S.J. Brodsky, G.P. Lepage, Phys. Lett B87 (1979) 359; Phys. Rev. D22 (1980) 2157.
- [7] N. Beisert and M. Staudacher, “The N=4 SYM integrable super spin chain,” Nucl. Phys. B **670** (2003) 439 doi:10.1016/j.nuclphysb.2003.08.015 [hep-th/0307042];
N. Beisert, Phys. Reports **405** (2005) 1, hep-th/0407277.
- [8] G. P. Korchemsky and E. Sokatchev, “Superconformal invariants for scattering amplitudes in N=4 SYM theory,” Nucl. Phys. B **839** (2010) 377 [arXiv:1002.4625 [hep-th]], J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, “*Dual superconformal symmetry of scattering amplitudes in N=4 super-Yang-Mills theory*”, Nucl. Phys. B **828** (2010) 317 [arXiv:0807.1095 [hep-th]].
- [9] J. M. Drummond, J. M. Henn and J. Plefka, “Yangian symmetry of scattering amplitudes in N=4 super Yang-Mills theory,” JHEP **0905** (2009) 046 [arXiv:0902.2987 [hep-th]];
J. M. Drummond and L. Ferro, “The Yangian origin of the Grassmannian integral,” JHEP **1012** (2010) 010 [arXiv:1002.4622 [hep-th]].
- [10] D. Chicherin and R. Kirschner, “Yangian symmetric correlators,” Nucl. Phys. B **877** (2013) 484 doi:10.1016/j.nuclphysb.2013.10.006 [arXiv:1306.0711 [math-ph]]
D. Chicherin, S. Derkachov and R. Kirschner, “Yang-Baxter operators and scattering amplitudes in N=4 super-Yang-Mills theory,” Nucl. Phys. B **881** (2014) 467 doi:10.1016/j.nuclphysb.2014.02.016 [arXiv:1309.5748 [hep-th]].
- [11] J. Broedel, M. de Leeuw and M. Rosso, “A dictionary between R-operators, on-shell graphs and Yangian algebras,” JHEP **1406** (2014) 170 doi:10.1007/JHEP06(2014)170 [arXiv:1403.3670 [hep-th]].
- [12] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, “Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz,” Commun. Math. Phys. **177** (1996) 381 doi:10.1007/BF02101898 [hep-th/9412229].
- [13] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, “*The S-Matrix in Twistor Space*”, JHEP **1003** (2010) 110 [arXiv:0903.2110 [hep-th]].
N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, “*A Duality For The S Matrix*”, JHEP **1003** (2010) 020 [arXiv:0907.5418 [hep-th]].
N. Arkani-Hamed, F. Cachazo and C. Cheung, “*The Grassmannian Origin Of Dual Superconformal Invariance*”, JHEP **1003** (2010) 036 [arXiv:0909.0483 [hep-th]].
- [14] L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka and M. Staudacher, “Harmonic R-matrices for Scattering Amplitudes and Spectral Regularization,” Phys. Rev. Lett. **110** (2013) no.12, 121602 doi:10.1103/PhysRevLett.110.121602 [arXiv:1212.0850 [hep-th]];
“Spectral Parameters for Scattering Amplitudes in N=4 Super Yang-Mills Theory,” JHEP **1401** (2014) 094 doi:10.1007/JHEP01(2014)094 [arXiv:1308.3494 [hep-th]].

- [15] L. Ferro, T. Lukowski and M. Staudacher, “ $\mathcal{N} = 4$ scattering amplitudes and the deformed Gramian,” Nucl. Phys. B **889** (2014) 192 doi:10.1016/j.nuclphysb.2014.10.012 [arXiv:1407.6736 [hep-th]].
- [16] N. Beisert, J. Broedel and M. Rosso, “On Yangian-invariant regularization of deformed on-shell diagrams in $\mathcal{N} = 4$ super-Yang-Mills theory,” J. Phys. A **47** (2014) 365402 doi:10.1088/1751-8113/47/36/365402 [arXiv:1401.7274 [hep-th]].
- [17] D. Müller, D. Robaschik, B. Geyer, F.M. Dittes, J. Horejsi, Fortschr. Physik 42 (1994) 101;
A.V. Radyushkin, Phys. Lett. B 385 (1996) 333;
X. Ji, Phys. Rev. D 55 (1997) 7714;
J.C. Collins, L. Frankfurt, M. Strikman, Phys. Rev. D 56 (1997) 2982.
- [18] P. Griffiths and J. Harris, “Principles of Algebraic Geometry,” Wiley Classics, 1994.
- [19] S. E. Derkachov, D. Karakhanyan and R. Kirschner, “Universal R-matrix as integral operator,” Nucl. Phys. B **618** (2001) 589 doi:10.1016/S0550-3213(01)00488-6 [nlin/0102024 [nlin-si]].
- [20] N. Beisert, “The complete one loop dilatation operator of $N=4$ superYang-Mills theory,” Nucl. Phys. B **676** (2004) 3 doi:10.1016/j.nuclphysb.2003.10.019 [hep-th/0307015].
- [21] S. E. Derkachov and R. Kirschner, “Parton interactions in the Bjorken limit of QCD,” Phys. Rev. D **64** (2001) 074013, doi:10.1103/PhysRevD.64.074013 [hep-ph/0101174].
D. Karakhanian and R. Kirschner, “Bjorken asymptotics of QCD and integrable systems,” In *Gurzadian, V.G. (ed.) et al.: From integrable models to gauge theories* 203-216
R. Kirschner, “Parton interactions in the Bjorken asymptotics,” 36th Winter school of St. Petersburg Inst. Nucl. Phys. hep-ph/0212248;
“Parton interaction in superYang-Mills theory,” JHEP **0407** (2004) 064, doi:10.1088/1126-6708/2004/07/064 [hep-th/0404060].
- [22] A. P. Bukhvostov, G. V. Frolov, L. N. Lipatov and E. A. Kuraev, “Evolution Equations for Quasi-Partonic Operators,” Nucl. Phys. B **258** (1985) 601. doi:10.1016/0550-3213(85)90628-5
A. P. Bukhvostov, E. A. Kuraev and L. N. Lipatov, “Deep Inelastic electron Scattering by a Polarized Target in Quantum Chromodynamics,” JETP Lett. **37** (1983) 482 [Pisma Zh. Eksp. Teor. Fiz. **37** (1983) 406] [Sov. Phys. JETP **60** (1984) 22] [Zh. Eksp. Teor. Fiz. **87** (1984) 37].
- [23] X. Artru and M. Mekhfi, “Transversely Polarized Parton Densities, their Evolution and their Measurement,” Z. Phys. C **45** (1990) 669. doi:10.1007/BF01556280
- [24] E. Sather and C. Schmidt, “Size and scaling of the double helicity flip hadronic structure function,” Phys. Rev. D **42** (1990) 1424. doi:10.1103/PhysRevD.42.1424
- [25] S. E. Derkachov, “Factorization of the R-matrix. I,” Zapiski POMI **335** (2006), 134-163 (J. Math. Sciences **143**(1) (2007), 2773-2790, math/0503396 [math-qa].
- [26] S. Derkachov, D. Karakhanyan and R. Kirschner, “Yang-Baxter R operators and parameter permutations”, Nucl. Phys. B **785** (2007) 263 [hep-th/0703076 [HEP-TH]].
D. Karakhanyan and R. Kirschner, “Jordan-Schwinger representations and factorised Yang-Baxter operators”, SIGMA **6** (2010) 029 doi:10.3842/SIGMA.2010.029 [arXiv:0910.5144 [hep-th]].
D. Chicherin, S. Derkachov, D. Karakhanyan and R. Kirschner, “Baxter operators for arbitrary spin”, Nucl. Phys. B **854** (2012) 393 doi:10.1016/j.nuclphysb.2011.08.029 [arXiv:1106.4991 [hep-th]].
- [27] L.N. Lipatov, “Small x physics in perturbative QCD,” Phys. Rept. **286** (1997) 131 doi:10.1016/S0370-1573(96)00045-2 [hep-ph/9610276].
- [28] S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, “Dual conformal symmetry on the light-cone,” Nucl. Phys. B **886** (2014) 1102 doi:10.1016/j.nuclphysb.2014.07.014 [arXiv:1306.5951 [hep-th]].
- [29] D. Karakhanyan, R. Kirschner and M. Mirumyan, “Universal R operator with deformed conformal symmetry,” Nucl. Phys. B **636** (2002) 529 doi:10.1016/S0550-3213(02)00350-4 [nlin/0111032 [nlin.SI]].
- [30] S. E. Derkachov, D. Karakhanian and R. Kirschner, “Heisenberg spin chains based on $sl(2-1)$ symmetry,” Nucl. Phys. B **583** (2000) 691 doi:10.1016/S0550-3213(00)00346-1 [nlin/0003029 [nlin-si]].